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GEODESICS AND VOLUMES IN REAL PROJECTIVE SPACES

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In 1951 P.M. Pu [11] proved the following result: Let P_2 be 2-dimensional real projective space, Γ the nontrivial free homotopy class of sectionally smooth ω: $[0, 1] \to P_2$, ω' the velocity vector of ω, h a Riemannian metric on P_2, $l_h = \inf\left\{\int_0^1 (h(\omega', \omega'))^{1/2} : \omega \in \Gamma\right\}$, and v_h the Riemannian volume of P_2 relative to h. Then $(l_h)^2/v_h \le \frac{1}{2}\pi$, with equality if and only if h has constant sectional curvature.

Pu's method was based on the fact that h is a conformal deformation of a Riemannian metric on P_2 of constant sectional curvature, that he could therefore average the metric over the group of isometries of P_2 with the Riemannian metric of constant sectional curvature, and then show than l_h increases and v_h decreases.

In this note we will consider two examples related to the appropriate (yet unsolved) generalization of Pu's result to higher dimensions. For convenience, we formulate our problem as a conjecture:

Pu's conjecture. Let P_n denote n-dimensional real projective space, Γ the nontrivial free homotopy class of continuous sectionally smooth ω: $[0, 1] \to P_n$, g the Riemannian metric of constant sectional curvature 1 on P_n, and h any Riemannian metric on P_n. Set $l_h = \inf\left\{\int_0^1 (h(\omega', \omega'))^{1/2} : \omega \in \Gamma\right\}$, where ω' is the velocity vector of ω, and v_h to be the Riemannian volume of P_n relative to h. Then

$$(l_h)^n/v_h \le (l_g)^n/v_g$$

with equality if and only if h has constant sectional curvature.

One easily sees that

$$(l_g)^n/v_g = \begin{cases} k!\pi^k, & n = 2k+1 , \\ (\pi/2)^k(2k-1)(2k-3)\cdots 3.1 , & n = 2k . \end{cases}$$

In § 1 we consider a 1-parameter family of Riemannian homogeneous metrics

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