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ON THE PRODUCT OF SCHUBERT CLASSES

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1. Introduction

1.1. In his paper [1] Kostant has described the generalized Schubert classes which serve as a basis of the cohomology ring of a large class of homogeneous spaces. The problem investigated here is that of determining the product of two Schubert classes as a linear combination of the others. The extensive notation needed to discuss this question is recalled in § 2. In § 3 some preliminary results are developed, and it is shown that it is sufficient to study the case of the generalized flag manifolds. § 4 contains the main result in which it is shown how the application of a certain linear operator to the product of two Schubert classes yields the product in terms of the other classes. § 5 contains some general statements about the products, including formulas applicable in some simple cases.

2. Background

2.1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let $\mathfrak{k} \subset \mathfrak{g}$ be a fixed compact real form of $\mathfrak{g}$. So $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is a real direct sum; and the Cartan-Killing form, denoted by $(\, , \,)$, is negative definite on $\mathfrak{k}$. This permits a $*$-operation to be defined on $\mathfrak{g}$ by $(x + iy)^* = -x + iy$ for $x, y \in \mathfrak{k}$. For any subspace $\mathfrak{s}$, $\mathfrak{s}^* = \{x^* \mid x \in \mathfrak{s}\}$.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a fixed Borel subalgebra. Then $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}^*$ is a Cartan subalgebra. Let $\Delta \subset \mathfrak{h}'$, the dual of $\mathfrak{h}$, be the set of roots associated with $\mathfrak{h}$. If $\mathfrak{m} = \{x \in \mathfrak{g} \mid (x, y) = 0 \ \forall y \in \mathfrak{b}\}$, then $\mathfrak{b} = \mathfrak{h} + \mathfrak{m}$ and $\mathfrak{g} = \mathfrak{b} + \mathfrak{m}^*$. Both $\mathfrak{m}$ and $\mathfrak{m}^*$ are maximal nilpotent subalgebras, and they are both $\mathfrak{h}$-modules under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Therefore $\mathfrak{m}$ is the complex span of $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{m})\}$ for a well-defined subset $\Delta(\mathfrak{m}) \subset \Delta$. Similary, $\mathfrak{m}^*$ is the span of $\{e_\varphi \mid \varphi \in \Delta(\mathfrak{m}^*)\}$. One can show that e_φ^* is a nonzero multiple of $e_{-\varphi}$, so that $\Delta(\mathfrak{m}^*) = -\Delta(\mathfrak{m})$; and one can describe a lexicographic ordering in $\mathfrak{h}'$ for which the positive roots $\Delta_+ = \Delta(\mathfrak{m})$ and the negative roots $\Delta_- = \Delta(\mathfrak{m}^*)$. Finally, one can normalize the root vectors $\{e_\varphi \mid \varphi \in \Delta\}$ so that both $(e_\varphi, e_{-\varphi}) = 1$ and $e_\varphi^* = e_{-\varphi}$ are satisfied. This is the normalization we shall assume hereafter. If $x_\varphi \in \mathfrak{h}$ denotes the root normal corresponding to the root φ, then the following product formulas hold:

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