J. DIFFERENTIAL GEOMETRY
8 (1973) 135-152

DIFFERENTIABLE FUNCTIONS ON BANACH SPACES WITH LIPSCHITZ DERIVATIVES

JOHN C. WELLS

Introduction

In this paper we study those functions in $C^k(E, F)$, (i.e., functions from two Banach spaces E to F having k continuous Frechet derivatives), whose k-th derivative is Lipschitz with constant M. On R^n we construct C^1 functions whose derivatives are piecewise linear with Lipschitz constant M. From this we obtain a Whitney type extension theorem for real-valued differentiable functions on Hilbert space, and show that every Hilbert space has C^1 partitions of unity. We examine the existence of "nontrivial" C^k functions with Lipschitz derivatives on separable Banach space and show that c_0 has no "nontrivial" C^1 function with Lipschitz derivative. We show that the Whitney extension theorem fails for separable Hilbert space by exhibiting a C^3 function on a closed subset of l^2 having no C^3 extension.

We make the definitions:

$$B^k_M(E, F) = \{f \mid f \in C^k(E, F) \text{ and } \|D^kf(y) - D^kf(x)\| \leq M\|x - y\| \text{ for all } x, y\}\,,$$
$$B^k(E, F) = \{f \mid f \in B^k_M(E, F) \text{ for some } M\}\,.$$

As in Bonic and Frampton [2] a Banach space E is said to be B^k smooth if there is a function $f \in B^k(E, R)$ with $f(0) \neq 0$ and support (f) bounded. Then B^{k+1} smoothness implies B^k smoothness, and E is said to be B^∞ smooth if E is B^k smooth for all k. We briefly summarize some results concerning C^k smoothness of separable Banach spaces. We refer to [2] and Eells [5] for more details.

1. Hilbert space is C^∞ smooth with C^∞ norm away from zero.
2. c_0 is C^∞ smooth with equivalent C^∞ norm away from zero. Kuiper.
3. A Lebesgue space $\mathscr{L}^p$ is C^∞ smooth for an even integer p, and C^{p-1} smooth but not D^p smooth for an odd integer p; Bonic and Frampton [2].
4. If E is separable, then E has a norm in $C^1(E - \{0\}, R)$ if and only if E^* is separable; Bonic and Reis [3].
5. Any C^k smooth separable Banach space has C^k partitions of unity; Bonic and Frampton [2].

In § 2 we prove some basic properties of $B^k_M(E, F)$, the most useful one being that $\{f \mid \|f\| \leq b$ on some open subset of $E\} \cap B^k_M(E, F)$ is closed in the

Communicated by J. Eells, Jr., February 11, 1972.