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CONGRUENCE OF HYPERSURFACES

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Perhaps the simplest type of function which measures curvature is the Frenet-Serret curvature of a plane curve. It is well-known that this curvature as a function of the arc length determines the congruence class of a curve. Its generalization in one direction is the normal curvature (or what we shall call the *bending*) of a hyperfurface. It may be defined as follows: let M be a hypersurface of dimension n in a Euclidean space $\boldsymbol{R}^{n+1}$, τ its tangent bundle, and

$$\alpha\colon \tau \times \tau \to \boldsymbol{R}$$

the second fundamental form. Let $\pi\colon G_1 \to M$ denote the Grassmann bundle of lines on M. Define the bending $K_\alpha\colon G_1 \to \boldsymbol{R}$ to be the function which assigns to each tangential direction v (at some point of M) the number

$$K_\alpha(v) = \alpha(u, v)/\|v\|^2 .$$

This function usually appears in textbooks only as an auxiliary before defining the sectional curvature. It is perhaps surprising that it has apparently not been noted before that this function essentially determines the congruence class of a hypersurface. To make this precise, we shall define two hypersurfaces M, $\overline{M}$ of $\boldsymbol{R}^{n+1}$ to be *similarly bent* if there exists a diffeomorphism $f\colon M \to \overline{M}$ such that $f^*K_{\bar\alpha} = K_\alpha$; in this case we shall call f a *bending preserving diffeomorphism*. We have

Theorem A. *Let M, $\overline{M}$ be two hypersurfaces in $\boldsymbol{R}^{n+1}$, $n \geq 2$, and $f\colon M \to \overline{M}$ a bending-preserving diffeomorphism. Suppose that*

a) *the nonumbilic points are dense in M, and*

b) *the sectional curvature of M is not identically zero.*

Then f is a congruence.

(Recall that a point $x \in M$ is *nonumbilic* if $K_{\alpha|\pi^{-1}(x)}$ is not identically constant.)

The congruence problem for hypersurfaces has a long history. The underlying analytic statement is that a diffeomorphism f, which is an isometry and preserves the second fundamental form (meaning $f^*\bar\alpha = \alpha$), is a congruence, and the point is to catch this analytic content in Frenet-Serret type, more intuitive geometric terms. A very interesting variant of this is Minkowski's

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