J. DIFFERENTIAL GEOMETRY
8 (1973) 75-84

SECOND ORDER CONNECTIONS. II

ROBERT H. BOWMAN

1. Introduction

The purpose of this paper is the development of certain implications of the second order connection, introduced previously by the present writer [1]. If M is an n-dimensional C^∞ manifold, we show that a linear second order connection on M determines a "covariant derivative" ∇' on TM, which satisfies the usual conditions over the ring $\mathfrak{F}'(TM)$, the vertical lift of the ring $\mathfrak{F}(M)$ of C^∞ functions on M. Using the properties of ∇', we obtain equations analogous to those of Gauss and Weingarten, and an analog of the second fundamental form.

If $A, B, C \in \mathfrak{X}'(TM)$, the module of C^∞ vector fields on TM over the ring $\mathfrak{F}'(TM)$, then we obtain the maps Tor (A, B) and $R(A, B)C$ which are $\mathfrak{F}'(TM)$ multilinear analogs of the torsion and curvature tensors. From the components of R we obtain equations analogous to those of Gauss and Codazzi, as well as an additional equation which defines a "vertical curvature tensor" on M. Finally, we obtain an invariant which we call the second order curvature of M; this yields as a special case the usual (first order) curvature of M.

2. Preliminary remarks

In this section we will briefly outline the main results of [1] utilized in the main body of this paper. The notation employed is essentially that of [1] and [2], with the summation convention employed on lower case Latin indices.

A second order connection on M is a connection on the bundle ${}^2_0\Pi : {}^2M \to M$ which naturally induces a (first order) connection on M. If ${}^1_0\Pi_*$ is the tangent map of ${}^1_0\Pi : TM \to M$, and $\tilde{D}$ is the connection map of the induced connection, then TTM and consequently ${}^2M$ may be given a vector bundle structure over M, such that if HTM and VTM are the horizontal and vertical subbundles of TTM determined by the vector bundle structure, then

$$ {}^1_0\Pi_* : HTM_p \to TM_{{}^1_0\Pi(p)} , \qquad \tilde{D} : VTM_p \to TM_{{}^1_0\Pi(p)} $$

are isomorphisms at each $p \in TM$.

Given a coordinate chart (U, ϕ) of M there are determined two sets of bases, relative to the induced coordinates $x^{01}, \cdots, x^{0n}; x^{11}, \cdots, x^{1n}$ on ${}^1_0\Pi^{-1}(U)$,

Communicated by K. Yano, August 23, 1971. Partially supported by an Arkansas State University research grant.