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SECOND ORDER CONNECTIONS

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1. Introduction

The purpose of this paper is to investigate certain higher order structures on an n-dimensional C^∞ manifold M, obtained by defining a connection on the bundle ${}^2_0\Pi\colon {}^2M \to M$. ${}^2M$ has been called the second extension of M by the present writer [1], and the second order tangent bundle by Yano and Ishihara [4].

We first define a second order connection on M as a connection on ${}^2_0\Pi\colon {}^2M \to M$ which induces a (first order) connection on M. It then follows that a second order connection of M defines a unique vector bundle structure on ${}^2_0\Pi\colon {}^2M \to M$ and thus allows us to define the connection map [2], [3] of a second order connection. Covariant differentiation of a section of the vector bundle ${}^2_0\Pi\colon {}^2M \to M$ with respect to a vector field on M is defined. This determines a concept of higher order parallelism and consequently two types of geodesics in M, which we call first and second order geodesics.

Given a (first order) connection on M we show that it induces a second order connection on M, and that if the first order connection is linear, then so is the induced second order connection. The case of a second order connection induced from a linear (first order) connection is investigated in detail. We show that in this case the first order geodesics are the usual geodesics (of the first order connection) of M, and that there are second order geodesics of M which are not first order geodesics. Using these second order geodesics we define a family of exponential maps and the related family of normal coordinates of M, which include the usual exponential map and normal coordinates as a special case.

We define the second order torsion tensor of a second order induced linear connection, and the second order Riemannian metric induced from a Riemannian metric on M. In the case that the first order connection is Riemannian we show that the induced second order connection is Riemannian with respect to the induced second order metric of the canonical first order metric. We define the second order curvature tensor of the induced linear second order connection, and the associated second order Riemann-Christoffel tensor, which may be used to define two natural invariants K_{I} and K_{II} which we call the Ist and IInd second order curvatures of M respectively.

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