J. DIFFERENTIAL GEOMETRY
7 (1972) 525-534

ISOPERIMETRIC INEQUALITIES FOR MANIFOLDS WITH BOUNDARY

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1. Introduction

It has long been known that if C is a simple closed curve of length L bounding a planar region of area A, then

$$L^2 - 4\pi A \geq 0 ,$$

where equality holds if and only if C is a circle. This is the classical isoperimetric inequality. In 1939 E. Schmidt [8] proved that if S is the $(n-1)$-dimensional measure of the boundary of a solid body M in Euclidean n-space, V is the n-dimensional measure of M, while σ and ν are the corresponding measures, respectively, for the n-ball of radius 1, then

$$(S/\sigma)^n - (V/\nu)^{n-1} \geq 0 ,$$

where equality holds if and only if M is an n-ball.

In 1959 W. T. Reid [7] generalized the classical isoperimetric inequality to regions on a surface. Suppose M is a C^2 image on a surface in Euclidean 3-space of a region in the plane bounded by a simple closed curve. Let A be the area of M, L the length of the boundary ∂M of M, H the mean curvature vector on M, and X the position vector to M. If the origin is an arbitrary point on ∂M, then

$$L^2 - 4\pi\left(A + \int_M X\cdot H d\nu\right) \geq 0 ,$$

where ν denotes 2-dimensional measure on M. In the case of equality, if the unit normal to M is constant on ∂M, then ∂M is a circle of radius $L/(2\pi)$. This result yields a new proof of a theorem originally due to T. Carleman [1]: If M is a minimal surface, then $L^2 - 4\pi A \geq 0$, where equality holds if and only if ∂M is a circle and M is the disk determined by ∂M.

Received April 12, 1971. Most of the results in this paper were obtained as a part of the author's doctoral dissertation written under the supervision of Professor Carl B. Allendoerfer at the University of Washington. The author is indebted to Professor Allendoerfer for suggesting the subject and for his further help and encouragement.