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GAUSSIAN CURVATURE AND CONFORMAL MAPPING

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1. Introduction and preliminaries

An ovaloid in E^3 is a closed convex surface of class C^2 with positive Gaussian curvature. We shall denote the unit sphere in E^3 by Σ.

L. Nirenberg has posed this question: given a C^∞ positive function K on Σ, do there exist an ovaloid S and a conformal mapping φ of S onto Σ so that the Gaussian curvature of S at P is equal to $K(\varphi(P))$ for all points P on S? In this paper we shall prove that the answer is yes in the special case where the given function K is even and sufficiently close to 1 pointwise. In fact, the ovaloid S will also turn out to be "almost spherical" in a natural sense. Some related questions will be discussed briefly in the last section of this paper.

We shall deal with the problem from the point of view of partial differential equations. A priori estimates for solutions and their derivatives with respect to local parameters on Σ will be needed. We shall therefore introduce now coodinate systems and function spaces appropriate to our problem. The unit sphere can be described by two overlapping coodinate patches. Say, we choose two overlapping open regions on Σ, one containing the north pole and bounded by a southern parallel of latitude, and one containing the south pole and bounded by a northern parallel of latitude. These two regions are mapped into the equator plane (x, y) through stereographic projection, the first one from the south pole and the second one from the north pole. Thus the parameter domains for the two spherical regions are two (coincident) open discs G_1 and G_2 in the equator plane. For the sake of definiteness, from now on we shall consistently use this parametrization of Σ, although our final results will be independent of the smooth parameters used to describe it.

The norm of a function of class C^k on Σ is defined as

$$\|f\|_k = \sum_{|\alpha|\le k} \sup_{G_1} |D^\alpha f| + \sum_{|\alpha|\le k} \sup_{G_2} |D^\alpha f| ,$$

where $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha f = \partial^{|\alpha|} f / \partial x^{\alpha_1} \partial y^{\alpha_2}$.

If, in addition, the kth derivatives of f, in each patch, satisfy a Hölder-condition with exponent α, $0 < \alpha < 1$, we define the norm of f to be

$$\|f\|_{k+\alpha} = \|f\|_k + h ,$$

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