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CLASSIFICATION OF THE SIMPLE SEPARABLE REAL L^*-ALGEBRAS

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Introduction

A *real* (complex) L^*-algebra is a Lie algebra L over the real (complex) numbers such that the underlying vector space is a Hilbert space (throughout this work the Hilbert space is assumed to be separable) and such that, for each $x \in L$, there is an $x^* \in L$ satisfying $([x, y], z) = (y, [x^*, z])$ for all y, z in L. L^*-subalgebras and L^*-ideals are defined in the usual way, with the additional property of being closed subspaces, invariant under the map $x \to x^*$. These algebras were introduced by J. R. Schue [11], [12], who obtained a complete classification of all simple separable complex L^*-algebras. V. K. Balachandran [1], [2], [3], [4], [5] gave more general settings to the techniques used by Schue for not necessarily separable L^*-algebras; he also defined the notions of real form and compact real form.

The main result of this work is the classification[1] of the simple separable real L^*-algebras up to L^*-automorphism.

We show in § 1 that the complexification $\tilde{L}$ of a simple real L^*-algebra is not simple if and only if $L = M^R$ (M^R denotes the real L^*-algebra obtained from M by restriction of scalars). Therefore, the classification reduces essentially, aside from simple real L^*-algebras having a complex structure which are in a one-to-one correspondence with the simple complex L^*-algebras, to the study of the real forms of all simple complex L^*-algebras.

If L is a real form of a semisimple L^*-algebra $\tilde{L}$, the decomposition $L = K + M$ (Hilbert direct sum), where $K = \{a \in L: a^* = -a\}$ and $M = \{a \in L: a^* = a\}$, defines an involutive L^*-automorphism S of L ($S|K = \mathrm{id}$ and $S|M = -\mathrm{id}$.) which can be extended to $\tilde{L}$ by linearity. S is called the involution of L associated to $\tilde{L}$. Conversely, if S is an involutive L^*-automorphism of L, then S leaves the unique compact form U (set of all self-adjoint elements of $\tilde{L}$) invariant and we have $U = K + iM$, the decomposition of U into eigenspaces of S. The real form $L = K + M$ is said to be associated to S.

There is a one-to-one correspondence between isomorphism classes of real

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[1] The classification was also obtained, independently, by Mr. Pierre de la Harpe.