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AN ANALYTIC OBSTRUCTION TO A COMPLEX TORAL ACTION ON A COMPLEX MANIFOLD

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Introduction

A theorem of R. Bott asserts that the Chern numbers of a compact complex manifold M can be computed from suitable invariants defined at the zeros of any nondegenerate holomorphic vector field on M. In particular, if M admits a nowhere vanishing holomorphic vector field, its Chern numbers vanish. Since one may view such a vector field as a holomorphic action of the complex numbers on M, we see that *if a compact complex manifold M admits a holomorphic action of a complex Lie group without fixed points, the Chern numbers of M must vanish.* For example, this is always the case if the group is compact. The purpose of the present note is to show that with this restriction on the action other cohomology classes must vanish too.

We will say a holomorphic vector-field X is *invariant* if it preserves a Hermitian structure g on M, i.e., if $L_X g = 0$. Note that we are assuming X is of type $(1, 0)$ so $X = Y - iJY$ where Y and JY are both infinitesimal isometries and J is the complex structure of M. Such a vector field arises in particular when M admits a holomorphic action of a complex torus. An invariant vector field is nowhere zero if either it arises from a torus action as above or the metric is Kaehlerian. The refined Chern classes of a complex manifold are certain cohomology classes in the refined de Rham cohomology ring of M which are defined from a Hermitian structure on M, but actually independent of the choice of structure; they generate a graded ring denoted $\widehat{ch}(M)$. By Hodge theory, the refined cohomology ring of a compact Kaehler manifold agrees with the usual de Rham cohomology ring.

In view of the above result of Bott, it is natural to expect that the existence of an invariant vector field on M has implications for the refined Chern ring. We state a result in this direction.

Theorem 1. *Suppose M is a Hermitian manifold admitting invariant vector fields $X_1, \cdots, X_k$ of type $(1, 0)$ which are linearly independent at some point $p \in M$. Then the refined Chern ring of M vanishes in dimensions exceeding $2(\dim_C M - k)$.*

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