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RIEMANNIAN MANIFOLDS WITH GEODESIC SYMMETRIES OF ORDER 3

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To S. S. Chern on his 60th birthday

1. Introduction

When is it possible to classify all compact simply connected Riemannian manifolds satisfying a given curvature condition? On the one hand Cartan [3], succeeded in classifying symmetric spaces. In contrast to this it seems hopeless, at least with our present knowledge, to classify compact simply connected Riemannian manifolds with parallel Ricci tensor. In the first place it is not known if such manifolds must be homogeneous (probably not); secondly, Wolf [22] has given an extremely large number of examples of compact homogeneous Einstein manifolds.

In the present paper it is shown that there is a different curvature condition for which a classification can be effected. Specifically, the following result is proved. Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on a differentiable manifold M.

Theorem (1.1). *Let M be a C^∞ compact simply connected almost Hermitian manifold with almost complex structure J, Riemannian connection ∇, and curvature tensor R. Assume that*

(*i*) $\nabla_X(J)X = 0$ *for all* $X \in \mathfrak{X}(M)$ (*that is, M is nearly Kählerian, see* § 2),

(*ii*) $\nabla_X(R)_{XJXXJX} = 0$ *for all* $X \in \mathfrak{X}(M)$.

Then M may be decomposed as a Riemannian product $M = M_1 \times \cdots \times M_r$ where $M_1, \cdots, M_r$ are listed in Tables V, VI, *and* VII, *in* § 6. *Furthermore if $M = M_1 \times \cdots \times M_r$ is such a product, then $M, M_1, \cdots, M_r$ satisfy* (*i*) *and* (*ii*), *and are Einstein manifolds.*

The basic idea behind this theorem involves the notion of *Riemannian* 3-*symmetric space*. Roughly, such a manifold is described as follows. For each point $p \in M$ there is an isometry $\theta_p : M \to M$ with p as an isolated fixed point such that $\theta_p^3 = 1$. Furthermore in § 4 we define *Riemannian locally* 3-*symmetric space*. Any such manifold has a naturally defined almost complex structure on it. See § 4 for precise definitions.

Theorem (1.1) is proved by giving formulas which characterize the curvature

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