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# COMPACT MANIFOLDS OF NONPOSITIVE CURVATURE

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## 0. Introduction and statement of results

Let $M$ be a compact $C^\infty$ riemannian manifold of nonpositive curvature[1] and with fundamental group $\pi$. It is well known [8, p. 102] that $M$ is a $K(\pi, 1)$ and thus completely determined up to homotopy type by $\pi$. In light of this fact it is natural to ask to what extent the riemannian structure of $M$ is determined by the structure of $\pi$, and the intent of this paper is to demonstrate that rather strong implications of this sort exist.

In the case that $M$ has strictly negative curvature, the group $\pi$ is known to be highly noncommutative. Every abelian, in fact, every solvable, subgroup of $\pi$ is cyclic [3]. It is therefore a plausible conjecture that in the nonpositive curvature case, $\pi$ will possess large amounts of commutativity only under special geometric circumstances. We shall show that this is true, that indeed those properties of $\pi$ which involve commutativity have a dramatic reflection in the riemannian structure of $M$.

Our first theorem concerns abelian subgroups of $\pi$, which, since no element of $\pi$ has finite order [8, p. 103], must be torsion free. As remarked above, when $M$ is negatively curved, every abelian subgroup has rank one. However, when the curvature of $M$ is simply nonpositive, we prove the following.

**The flat torus theorem.** *There exists an abelian subgroup of rank $k$ in $\pi$ if and only if there exists a flat $k$-torus isometrically and totally geodesically immersed in $M$.*

The second theorem concerns the case where $\pi$ is a product of groups. In particular, we shall prove:

**The splitting theorem.** *Let $M$ be real analytic and assume that $\pi$ has no center. If $\pi$ can be expressed as a direct product of groups $\pi = \mathscr{A}_1 \times \cdots \times \mathscr{A}_N$, then $M$ is isometric to a riemannian product $M = M_1 \times \cdots \times M_N$, where $\pi_1(M_k) = \mathscr{A}_k$ for $k = 1, \cdots, N$.*

It is shown in § 4 that in the case that $\pi$ has a nontrivial center, the splitting theorem, as stated, is not true. However, by a slight weakening of the conclusion, one can obtain a similar theorem for the general case.

As one may by now suspect, the appearance of a nontrivial center in $\pi$ must

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[1] Throughout the paper curvature refers to sectional curvature.