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ON THE MATHEMATICAL FOUNDATIONS OF ELECTRICAL CIRCUIT THEORY

S. SMALE

The goal of this note is to derive the differential equations for simple (non-linear) electrical circuits with resistors, inductors and capacitors. I would like to express my deep indebtedness in these matters to George Oster. Oster supplied me with the basic references to the literature, and conversations on this subject with him were very helpful. The reference Brayton & Moser [4] was also very helpful.

1. A simple electrical circuit provides us first of all with an oriented graph G which will be assumed to be connected but not necessarily planar. This is a one-dimensional cell complex with branches or elements (1-cells) and nodes (0-cells or vertices).

The states of the circuit have two components, the currents through the branches and the voltages across the branches. Let C_j be the vector space of real j-chains of G, C^j the j-cochains of G, $j = 0, 1$. Thus C^j can be thought of as the dual vector space of C_j. As is well-known, the currents in the circuit can be represented as an element i of C_1. Thus $i = \sum_\sigma i_\sigma c_\sigma$ where σ ranges over all the branches, c_σ is the σ-th branch and i_σ is the current through the σ-th branch.

The voltages in the circuit can be represented by an element v of C^1. Thus $v = \sum_\sigma v_\sigma c'_\sigma$ where v_σ is the voltage across the σ-th branch and c'_σ is the cochain which is 1 on c_σ, and 0 on the others.

Let $\mathscr{S} = C_1 \times C^1$. Then $s = (i, v) \in \mathscr{S}$ is a state (unrestricted) of the circuit. Physical laws (Kirchhoff and Generalized Ohm) will constrain the physical states to lie in a submanifold $\sum$ of $\mathscr{S}$ which we proceed to define.

Denote the boundary map by $\partial\colon C_1 \to C_0$ and coboundary by $\partial^*\colon C^0 \to C^1$. Thus ∂ is a linear transformation of vector spaces, and ∂^* is its adjoint on the dual spaces.

In this context the Kirchhoff laws KCL, KVL can be expressed as follows (see Branin [2] and the references therein):

$$KCL\colon i \in \operatorname{Ker} \partial\,, \qquad KVL\colon v \in \operatorname{Image} \partial^*\,.$$

The first condition just expresses the fact that the currents entering a node

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