J. DIFFERENTIAL GEOMETRY
6 (1972) 411-419

A FORMULA FOR THE RADIAL PART OF THE LAPLACE-BELTRAMI OPERATOR

SIGURDUR HELGASON

Let V be a manifold and H a Lie transformation group of V. Suppose $Du = 0$ is a differential equation on V, both the differential operator D and the function u assumed invariant under H. Then the differential equation will involve several inessential variables, a fact which may render general results about differential operators rather ineffective for the differential equation at hand. Thus although D may not be an elliptic operator it might become one after the inessential variables are eliminated (cf. [3, p. 99]).

This viewpoint leads to the general definition (cf. [7]) of the transversal part and radial part of a differential operator on V given in §§ 2 and 3. The radial part has been constructed for many special differential operators in the literature; see for example [1], [3], [4], [5], [8] for Lie groups, Lie algebras and symmetric spaces, [9], [6] for some Lorentzian manifolds. Our main result, formula (3.3) in Theorem 3.2, includes various known examples worked out by computations suited for each individual case. See Harish-Chandra [4, p. 99] for the Laplacian on a semisimple Lie algebra, Berezin [1] and Harish-Chandra [3, § 8] for the Laplacian on a semisimple Lie group, and Harish-Chandra [5, § 7] and Karpelevič [8, § 15] for the Laplacian on a symmetric space. The author is indebted to J. Lepowsky for useful critical remarks.

Notation. If V is a manifold and $v \in V$, then the tangent space to V at v will be denoted V_v; the differential of a differentiable mapping φ of one manifold into another is denoted $d\varphi$. We shall use Schwartz' notation $\mathscr{E}(V)$ (resp. $\mathscr{D}(V)$) for the space of complex-valued C^∞ functions (resp. C^∞ functions of compact support) on V. Composition of differential operators D_1, D_2 is denoted $D_1 \circ D_2$.

2. The transversal part of a differential operator

Let V be a manifold satisfying the second axiom of countability, and H a Lie transformation group of V. If $h \in H$, $v \in V$, let $h \cdot v$ denote the image of v under H and let H^v denote the isotropy subgroup of H at v. Let $\mathfrak{h}$ denote the Lie algebra of H. If $X \in \mathfrak{h}$, let X^+ denote the vector field on V *induced by* X, i.e.,

Received November 3, 1971. Supported in part by National Science Foundation Grant GP-22928.