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PROLONGATIONS AND COMPLETIONS OF RIEMANNIAN MANIFOLDS

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A *Riemannian prolongation* of a connected Riemannian manifold M is a connected Riemannian manifold N of the same dimension, along with a proper isometric imbedding of M in N. If N is complete, the prolongation is called a *Riemannian completion*.

In the first section, we give some examples and formulate some obstructions to prolongation at a point of the Cauchy boundary of M. The examples show that there may not exist maximal subsets of the Cauchy boundary on which M can be prolonged, and that M may admit a prolongation on its whole Cauchy boundary whithout admitting a Riemannian completion. The main theorem of § 2 states that for any Riemannian manifold there is a nonprolongable metric with the same Cauchy sequences in the class of conformally equivalent metrics. In § 3, a theorem for conformally equivalent complete metrics is proved, which gives a sufficient condition for the existence of Riemannian completions.

1. Prolongations

A Riemannian prolongation determines a *metric prolongation* of the underlying metric space, and a Riemannian completion determines, by the Hopf-Rinow theorem, a *metric completion*. Specifically, by a metric prolongation of a metric space M we mean a metric space N and a proper distance-nonincreasing injection of M in N, with N complete giving a metric completion. For example, the Cauchy completion CM consists of equivalence classes under asymptotism of Cauchy sequences in M, with the obvious metric and embedding. We denote by $\dot{M}$ the *Cauchy boundary* CM-M. A metric prolongation of M to N is said to *prolong on a subset* S of $\dot{M}$ if the Cauchy sequences which determine S converge in N. Note that we do not require the corresponding map of S into N to be injective; indeed, for Riemannian prolongations the geometry of M may force identifications on $\dot{M}$ (see Example 1). S will be called a *prolongation set*; *the* prolongation set of the prolongation will be the largest such S.

From now on, prolongation will mean Riemannian prolongation. We can make several simple observations. For a Riemannian manifold, CM need not

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