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COMPACT FLAT RIEMANNIAN MANIFOLDS

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Let M be a compact flat Riemannian manifold of dimension n, and π its fundamental group. Then we have the following theorem of Bieberbach-Auslander-Kuranishi [1], [2]:

Theorem 1. *The group π is torsion free and satisfies the following exact sequence*

$$1 \to A \to \pi \to \Phi \to 1$$

where A is a finitely generated maximal abelian subgroup of π, and Φ is a finite group. Conversely, every group with the above property is the fundamental group of a compact flat Riemannian manifold of dimension n. The group Φ is the holonomy group of M.

In [3], E. Calabi announced that every compact flat Riemannian manifold with nonzero first betti number can be given by a construction which we shall call the Calabi construction. The purpose of § 1 of this paper is to generalize Calabi's theorem to the case where M has positive semidefinite Ricci tensor and to study the condition under which the Calabi construction is possible. We show that if Φ is cyclic or if the dimension of M is odd and Φ is of odd order, then the first betti number of M is not zero. This will follow from a fixed point theorem.

In [9], A. T. Vasquez proved the following.

Theorem 2. *There is associated with every finite group Φ a positive integer $n(\Phi)$ such that: if M is a compact flat Riemannian manifold with holonomy group Φ, and $\dim M \geq n(\Phi)$, then M is a flat toral extension of another flat manifold of dimension $\leq n(\Phi)$.*

The integer $n(\Phi)$ is not known except for the special case when Φ is a prime order group for which $n(\Phi) = 1$ and when Φ is $Z_2 \times Z_2$ for which $n(\Phi) \leq 6$, cf. [8]. In § 2 of this paper we prove that $n(\Phi)$ can be chosen to be less than or equal to the sum of the indices of maximal cyclic subgroups of Φ. When Φ is of prime order or is $Z_2 \times Z_2$, we obtain the bound stated above. Theorem 2 is reproved by using some elementary methods and hence avoiding results of I. Reiner on integral representation of prime order groups and homology of groups.

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