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COERCIVENESS IN THE NEUMANN PROBLEM

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Let M' be a Riemannian manifold of dimension n, let E, F, and G be Hermitian vector bundles over M', and let

$$(1) \qquad E \xrightarrow{A} F \xrightarrow{B} G$$

be a complex of first order linear partial differential operators. For each complex cotangent vector $\zeta \in T_x^*(M') \otimes \boldsymbol{C}$ let

$$(2) \qquad E_x \xrightarrow{a(\zeta)} F_x \xrightarrow{b(\zeta)} G_x$$

denote the sequence of symbol maps corresponding to (1). We shall assume that (2) is exact for all real $\zeta \neq 0$ so that (1) is an elliptic complex.

Now let M be a compact n-dimensional manifold-with-boundary which is smoothly imbedded in M', and let r be a real C^∞ function on M' such that $M = \{x \in M' \mid r(x) \geq 0\}$ and $\partial M = \{x \in M' \mid r(x) = 0\}$ and such that dr never vanishes on ∂M. Our purpose here is to discuss the coercive estimate

$$(3) \qquad \|u\|_1 \leq c\{\|A^*u\| + \|Bu\| + \|u\|\}$$

for $u \in C^\infty(M, F)$ satisfying the boundary condition

$$(4) \qquad a(dr)^*u = 0 \quad \text{on} \quad \partial M \ ,$$

where $\|\cdot\|$ denotes the L_2 norm for sections defined over M, and $\|\cdot\|_1$ denotes any of the equivalent norms on the Sobolev space $\mathscr{H}_1(M, F)$. We prove (Theorem 3) that if the coercive estimate holds for all compact $M \subset M'$, and (1) is part of a Spencer complex, then (1) is locally exact; we also show (Theorem 4) that if (2) is exact for all complex $\zeta \neq 0$, then the coercive estimate (3) holds for all compact M. Under additional assumptions (Theorem 5) we give explicit necessary and sufficient conditions for (3).

Some of our results here (Theorems 1 and 5, in particular) were suggested by recent work of V. W. Guillemin and S. Sternberg [4]. The author is also indebted to J. J. Kohn for several helpful conversations.

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