J. DIFFERENTIAL GEOMETRY
6 (1972) 343-355

A GENERALIZATION OF KAEHLER GEOMETRY

S. I. GOLDBERG

1. Introduction

In this paper a class of non-Kaehler manifolds is introduced which by its very definition is included in the generalization of Kaehler geometry given by Chern [1] (see also Weil [8]). This class is of particular interest because of its additional structure thereby yielding in the compact case topological consequences of special interest. The spaces considered are the globally framed f-manifolds $M(f, E_a, g)$, $a = 1, \cdots, 2n - r$, where $\dim M = 2n$ is even and rank $f = r$, previously studied by Yano and the author in [2]–[4]. Thus, it is necessary that the structural group of the tangent bundle of M can be reduced to thc direct product of $U(r/2)$ and $O(m - r)$, the unitary group in $r/2$ complex variables and the orthogonal group in $m - r$ variables. In [3], the structure tensors f and the E_a are assumed to be parallel fields with respect to the Riemannian connection, but since this implies that there is an underlying Kaehlerian structure the theory is not a satisfactory one. The proper generalization along these lines is provided by assuming (a) the fundamental form F of the f-structure is closed, (b) the Nijenhuis torsion of f vanishes, and (c) the field f is parallel along the integral curves of the vector fields E_a. Conditions (a)–(c) are clearly satisfied if (a) is replaced by the stronger condition that f be a parallel field and, in fact, they are equivalent to the latter (Theorem 1, Corollary 2). When $r = m$, the f-structure of M is Kaehlerian.

Chern's generalization of Kaehlerian geometry may be described as follows. Suppose that the structure group of the tangent bundle of a real C^∞ manifold of dimension m is reducible to a subgroup G of the rotation group in m variables. (Observe that $U(r/2) \times O(m - r) \subset O(m)$.) A connection can be defined with the group G. The vanishing of torsion of this connection is then a natural generalization of the Kaehler property. This includes the generalization due to Lichnerowicz [6], namely the even dimensional orientable Riemannian manifolds carrying a 2-form, of maximal rank everywhere, whose covariant derivative vanishes.

Conditions (a) and (b) are analogous to those characterizing Kaehler manifolds, whereas (c) is required when the rank of f is less than $2n$, and otherwise is vacuous. The f-manifold has an associated Kaehler structure if and only if

Received March 13, 1971 and, in revised form, September 10, 1971. Research supported by the National Science Foundation.