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ON THE GEOMETRY AND CLASSIFICATION OF ABSOLUTE PARALLELISMS. I

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1. Introduction and summary

A. General introduction

Within the context of riemannian geometry, the euclidean spaces are distinguished as the only complete simply connected manifolds in which parallel translation of tangent vectors is independent of path. When Élie Cartan developed the general notion of affine connection he saw that this absolute sort of parallelism was (at least locally) a matter of vanishing curvature. Then Cartan and Schouten [3] described curvature-free connections on Lie groups, thus exhibiting absolute parallelisms on group manifolds. This generalized Clifford's parallelism on the 3-sphere, which had previously been an isolated phenomenon. Cartan and Schouten [4] also gave a local description of the riemannian manifolds which have an absolute parallelism whose parallel vector fields have constant length and integrate to geodesics; they are the products of euclidean spaces, compact simple groups and 7-spheres. Unfortunately their reduction to the irreducible case may have gaps, and the cause of the parallelism on the 7-sphere was not too clear.

Here we extend the work of Cartan and Schouten to pseudo-riemannian manifolds. This means that the metric form ds^2 is of some nondegenerate signature[1] (p, q), but not necessarily of positive definite signature $(n, 0)$. The de Rham decomposition theorem fails for indefinite signatures of metric; in fact, our example (3.7) shows that it fails for bi-invariant pseudo-riemannian metrics on nilpotent Lie groups. Thus we adopt an algebraic curvature condition ("reductive type"; see (5.7) below) which is automatic in the riemannian case and ensures us of a de Rham decomposition. Our main results are proved under that condition.

Let (M, ds^2) be a connected pseudo-riemannian manifold of "reductive type" with an absolute parallelism ϕ which satisfies the Cartan-Schouten consistency conditions described above. Our main result (Theorem 9.1) says that (M, ds^2)

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[1] Signature (p, q) means p positive squares and q negative squares in dimension $p + q$, as $\sum_1^p (x^i)^2 - \sum_1^q (x^{i+p})^2$ on R^{p+q}.