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AUTOMORPHISMS AND INTEGRABILITY OF PLANE FIELDS

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A p-plane field on an n-dimensional manifold is a section in the bundle associated to the tangent bundle with fiber the Grassmann manifold of p-planes in affine space $\boldsymbol{R}^n$. It is integrable if each point has a neighborhood U homeomorphic to affine space in such a way that the restriction of the plane field to U is carried by the induced tangent map onto a field of parallel planes. Since a field of parallel planes in $\boldsymbol{R}^n$ is preserved by any translation, the restriction to U of the field admits a transitive abelian group of automorphisms, that is, homeomorphisms such that their tangent maps take the field onto itself. In this paper, we shall prove the converse.

Theorem. *A p-plane field is integrable if and only if each point has a neighborhood homeomorphic to affine space on which the restriction of the field admits a transitive abelian group of automorphisms. The homeomorphisms occurring in the definition of integrability and in the automorphism groups are of the same class* C^k *for some* $k = 0, 1, \cdots, \infty$.

This theorem follows immediately from the preceding remarks and the following lemma:

Lemma 1. *Let G be a transitive abelian subgroup of the group of homeomorphisms of class* C^k *of* $\boldsymbol{R}^n$, *where* $k = 0, 1, \cdots, \infty$. *Then G is conjugate to the group of translations, and the conjugating element is unique up to an affine map.*

Indeed, suppose the lemma holds. Let $f\colon U \to \boldsymbol{R}^n$ be a homeomorphism, G_1 be a transitive abelian group of automorphisms of the restriction of the field to U, and T be the group of translations in $\boldsymbol{R}^n$. Then there exists a homeomorphism g of $\boldsymbol{R}^n$ such that

$$gfG_1f^{-1}g^{-1} = T .$$

Hence the tangent map induced by gf takes the given p-plane field into one preserved by the translation group of $\boldsymbol{R}^n$, that is, a parallel field. Hence the p-plane field is integrable as required.

It remains to prove Lemma 1. The idea of the proof is to topologize the given

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