J. DIFFERENTIAL GEOMETRY
6 (1971) 259-262

REMARKS ON CURVATURE AND THE EULER INTEGRAND

ALAN WEINSTEIN

1. Review of the problem

We define an *n-dimensional curvature tensor* as an n^4-tuple $R = (R_{ijkl})$, $1 \leq i, j, k, l \leq n$, of real numbers satisfying the symmetry relations

$$R_{ijkl} = -R_{jikl} = R_{klij} \, , \tag{1}$$

and

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \, , \tag{2}$$

and we denote the vector space of all such curvature tensors by K^n.

The polynomial function $\chi^{2n}\colon K^{2n} \to \boldsymbol{R}$ defined by the formula

$$\chi^{2n}(R) = (-1)^n \sum_{i,j} \varepsilon_{i_1 \cdots i_{2n}} \varepsilon_{j_1 \cdots j_{2n}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2n-1} i_{2n} j_{2n-1} j_{2n}}$$

will be called the Euler integrand in dimension $2n$ since, by the generalized Gauss-Bonnet theorem, the Euler characteristic of an oriented riemannian manifold M of dimension $2n$ is obtained, up to a positive constant, by evaluating χ^{2n} on the components of the curvature tensor in orthonormal frames and integrating the resulting real valued function over M, using the volume element associated with the given riemannian metric.

It has been conjectured by H. Hopf that the Euler characteristic of an even dimensional riemannian manifold with positive sectional curvature is positive, and it may even be the case that the Euler integrand is positive in this situation. The present note is devoted to the presentation of some remarks on this question. We continue by fixing some more terminology.

The polynomial function $\sigma^n\colon K^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ defined by the formula

$$\sigma^n(R, x, y) = -\sum R_{ijkl} x_i y_j x_k y_l$$

may be called the sectional curvature function, since $\sigma^n(R, x, y)$ is, up to a positive constant, the sectional curvature of the plane spanned by x and y, computed from the curvature tensor R. Of course, x and y really span a plane

Received November 14, 1970, and, in revised form, January 21, 1971.