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THE SPLITTING THEOREM FOR MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

JEFF CHEEGER & DETLEF GROMOLL

The purpose of this paper is to extend Toponogov's splitting theorem [4], [7] for manifolds of nonnegative sectional curvature to manifolds of nonnegative Ricci curvature. By use of the extension we are able to show that our results on the structure of the fundamental group in the compact case and on locally homogeneous spaces, proved in [4] for manifolds of nonnegative sectional curvature, remain valid for manifolds of nonnegative Ricci curvature. In addition, we sharpen a result of Milnor on the rate of growth of the fundamental group in the noncompact case. As a final application, we show under fairly general circumstances (in particular, if M is locally irreducible) that the holonomy group of an arbitrary compact riemannian manifold is compact. By way of explanation, we remark that Berger has shown that the holonomy group of M is compact if M is locally irreducible and the Ricci curvature of M does *not* vanish identically. The case $\mathrm{Ric}_M \equiv 0$ is precisely the one we are able to handle. The last application was suggested during a conversation between the authors and L. Charlap.

Let M be a complete riemannian manifold. Recall that a ray (respectively a line) in M is a geodesic $\gamma\colon [0, \infty) \to M$ (respectively $\gamma\colon (-\infty, \infty) \to M$) each segment of which is minimal. With each ray γ in M we associate a function g_γ as follows: Let $g_t(x) = \overline{x, \gamma(t)} - t$ for $t \geq 0$ where the bar denotes metric distance. The function g_t is continuous, but not differentiable on the cut locus of $\gamma(t)$. It follows easily from the triangle inequality that the family g_t is uniformly equicontinuous. For fixed x, the function $t \to g_t(x)$ is decreasing on $[0, \infty)$ and bounded below by $-\overline{x, \gamma(0)}$. Hence, for $t \to \infty, g_t$ converges uniformly on compact sets to a continuous function g_γ.

Theorem 1. *If M has nonnegative Ricci curvature, then for any ray γ in M the function g_γ is superharmonic.*

Here superharmonic means that given any connected compact region D in M with smooth boundary ∂D, one has $g_\gamma \geq h_\gamma$ on D where h_γ is the continuous function on D which is harmonic on $\operatorname{int} D$ with $h_\gamma | \partial D = g_\gamma | \partial D$. Since this is true for all connected domains, a standard argument gives that if $h_\gamma(y) = g_\gamma(y)$ for $y \in \operatorname{int} D$, then $g_\gamma \equiv h_\gamma$ on D. If, moreover, the sectional curvature of M is

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