J. DIFFERENTIAL GEOMETRY
5 (1971) 503-515

# THIRD ORDER NONDEGENERATE HOMOTOPIES OF SPACE CURVES

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## 1. Introduction

A space curve given by a $C^3$ immersion $X: S^1 \to E^3$ of the unit circle $S^1$ into a Euclidean 3-space $E^3$ is said to be third order nondegenerate if it has nonvanishing torsion, and of course nonvanishing curvature. Examples of such curves are coiled springs which are joined into closed curves; some telephone cords are made this way. Two such space curves are third order nondegenerately homotopic if they are connected by a 1-parameter family of such curves. (See Feldman [1] for the general definition of a $p$th order nondegenerate map between manifolds of arbitrary dimensions.)

**Theorem 1.** *There are four third order nondegenerate homotopy classes of space curves.*

Let $e_1$ be the unit tangent vector of the curve $X$, respecting the orientation when that is prescribed. The spherical image or tangential indicatrix is given by the map $e_1: S^1 \to S^2$, where $S^2$ is the unit 2-sphere in $E^3$. It is easy to check that because the torsion of the curve $X$ never vanishes the geodesic curvature of its spherical image is never zero. Furthermore, because the curve is closed, the spherical image must cross every great circle, or what is the same, contain the center of $S^2$ in the interior of its convex hull; this observation about closed space curves is due to C. Loewner (see Fenchel [3]). Thus a 1-parameter family of space curves each with nonzero torsion gives rise to a 1-parameter family of spherical curves, each of which has nonzero geodesic curvature and which contains the center of $S^2$ in its convex hull.

In [5] we have classified second order nondegenerate homotopy classes of curves on the unit 2-sphere; a second order nondegenerate curve on $S^2$ being one such that the geodesic curvature is not zero. Two nondegenerate curves are nondegenerately homotopic if and only if they are regularly homotopic, their geodesic curvatures have the same sign and they are either both simple or both have double points. Nondegenerate simple curves must lie in a hemisphere (see Fenchel [3]), but among the classes of curves with double points it is possible to find nondegenerate curves which cross every great circle. Representatives of the four second order nondegenerate homotopy classes of

Received October 1, 1970 and, in revised form, January 21, 1971. Research was supported by a National Science Foundation grant GP 11533.