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KÄHLER SURFACES OF NONNEGATIVE CURVATURE

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1. A well-known theorem of Andreotti and Frankel [4] asserts that any compact Kähler surface of positive sectional curvature is biholomorphically equivalent to the complex projective plane. In this paper we investigate compact complex analytic surfaces which carry a Kähler metric of nonnegative curvature. Our basic assumption is ostensibly weaker than that of nonnegative sectional curvature, and invokes the notion of *holomorphic bisectional curvature* recently introduced by Goldberg and Kobayashi [5]. If p and p' are planes in the tangent space of a Kähler manifold each invariant with respect to the almost-complex structure tensor J, then the *(holomorphic) bisectional curvature* $H(p,p')$ is defined by

$$H(p,p') = R(X,JX,Y,JY) \ , \tag{1}$$

where R is the Riemann curvature tensor, and X and Y are unit vectors in the planes p and p'. From Bianchi's identity,

$$H(p,p') = R(X,Y,X,Y) + R(X,JY,X,JY) \ , \tag{2}$$

so that $H(p,p')$ is the sum of two sectional curvatures. It follows that nonnegative sectional curvature at a point implies nonnegative bisectional curvature at that point. (For complex dimension 2, it follows from the results of this paper that everywhere nonnegative bisectional curvature is equivalent to everywhere nonnegative sectional curvature.) With this definition, we may state our main result.

Theorem. *Let M be a compact Kähler surface whose holomorphic bisectional curvature is everywhere nonnegative. Then one of the following holds:*

(i) *M is biholomorphically equivalent to the complex projective plane $\boldsymbol{P}^2$.*

(ii) *M is biholomorphically equivalent to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, and the metric is a product of metrics of nonnegative curvature.*

(iii) *M is flat.*

(iv) *M is a ruled surface* (i.e., *$\boldsymbol{P}^1$-bundle*) *over an elliptic curve. In this case, the universal covering space of M is $\boldsymbol{C} \times \boldsymbol{P}^1$ endowed with the product of the flat metric on $\boldsymbol{C}$ and a metric of nonnegative curvature on $\boldsymbol{P}^1$.*

In § 3 we show that if the Ricci tensor is nondegenerate at any point, then

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