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THE AXIOM OF SPHERES IN RIEMANNIAN GEOMETRY

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In his book on Riemannian geometry [1] Elie Cartan defined the axiom of r-planes as follows. A Riemannian manifold M of dimension $n \geq 3$ satisfies the *axiom of r-planes*, where r is a fixed integer $2 \leq r < n$, if for each point p of M and any r-dimensional subspace S of the tangent space $T_p(M)$ there exists an r-dimensional totally geodesic submanifold V containing p such that $T_p(V) = S$. He proved that if M satisfies the axiom of r-planes for some r, then M has constant sectional curvature [1, § 211].

We propose

Axiom of r-spheres. *For each point p of M and any r-dimensional subspace S of $T_p(M)$, there exists an r-dimensional umbilical submanifold V with parallel mean curvature vector field such that $p \in V$ and $T_p(V) = S$.*

We shall prove

Theorem. *If a Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r < n$, then M has constant sectional curvature.*

The special case where $r = n - 1$ was proved by J. A. Schouten (see [3, p. 180]). In this case the condition of parallel mean curvature vector field simply means constancy of the mean curvature.

1. Preliminaries

Let M be a Riemannian manifold of class C^∞, and let V be a submanifold. The Riemannian connections of M and V are denoted by ∇ and ∇', respectively, whereas the normal connection (in the normal bundle of V in M) is denoted by $\nabla^\perp$. The second fundamental form α is defined by

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y) ,$$

where X and Y are vector fields tangent to V. On the other hand, for any vector field ξ normal to V, the tensor field A_ξ of type (1,1) on V is given by

$$\nabla_X \xi = -A_\xi(X) + \nabla^\perp_X \xi ,$$

where X is a vector field tangent to V. We have

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