J. DIFFERENTIAL GEOMETRY
5 (1971) 437-449

ALGEBRAS OF MATRICES UNDER DEFORMATION

W. STEPHEN PIPER

1. Introduction

The subject of this discussion is families of one-parameter deformations of the associative algebras of $n \times n$ upper triangular real matrices; the purpose is to expand the set of examples of algebraic deformations. Gerstenhaber [1] has given an example of a commutative associative algebra which when deformed is non-commutative. Also, a large class of associative algebras A, namely the class of semi-simple algebras, which includes the algebras of $n \times n$ matrices, has the second Hochschild cohomology group $H^2(A, A)$ equal to zero. These algebras are rigid, meaning that their only deformations are trivial, that is, equivalent to those generated by vector space isomorphisms.

We consider the algebras A_n of $n \times n$ upper triangular real matrices having equal diagonal elements. For any $n \geq 2$, $\dim Z^2(A_n, A_n) > \dim B^2(A_n, A_n)$, and hence $H^2(A_n, A_n) \neq 0$ (§ 4). In the case of $n = 3$, we exhibit 2-cocycles which can not be integrated to a deformation of A_3. Although $H^3(A_2, A_2) \neq 0$, we prove that any infinitesimal deformation f of A_2 and any partial integration of f can be completed to a deformation of A_2. In other words, all obstructions to the integration of f vanish, and as we shall see, with restriction only on the choice of four of the eight coefficients for the cochains involved.

§ 2 presents a brief review of the definitions in algebraic deformation theory, and § 3 introduces the terminology which proves useful in analysis of the deformations of A_n. The existence of non-trivial infinitesimal deformations of A_n is proven in § 4, together with the fact that $H^3(A_n, A_n) \neq 0$. The particular cases of $n = 2$ and 3 are taken up in §§ 5 and 6. Formula 19 and § 7 provide examples of deformations of $A_n, n \geq 2$.

2. Background

We recall from [1] and [2] the principal definitions of algebraic deformation theory. Given an associative algebra A with multiplication denoted by juxtaposition, we define a (one-parameter) deformation of A to be a formal power series,

$$(1) \qquad F_t(\alpha, \beta) = \alpha\beta + f_1(\alpha, \beta)t + f_2(\alpha, \beta)t^2 + \cdots , \qquad \alpha, \beta \in A ,$$

Communicated by D. C. Spencer, October 6, 1970.