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APPLICATION OF THE HIGHER OSCULATING SPACES TO THE SPHERICAL PRINCIPAL SERIES

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1. Introduction

The purpose of this paper is to use a geometric construction (analogous to the higher osculating spaces and fundamental forms of immersions) to study certain infinite dimensional Banach representations of semisimple Lie groups (the spherical principal series). Few of our results are new, most can be found in Kostant [5] or Helgason [4]. However, the proofs are new and quite elementary (in comparison to those of Kostant and Helgason).

In § 2 we define the spherical principal series and study duality in the series. In § 3 we study cyclic vectors for these representations and prove several results on finite dimensional class 1 representations including the fact that every finite dimensional class 1 representation is realized as a canonically defined subspace of a principal series representation. In § 4 the geometric construction alluded to above is given. § 5 is devoted to applications of the results of §§ 1–4. We prove in particular that almost all of the elements of the spherical principal series (not necessarily unitary) are irreducible. This is the weakest possible way of stating the result of Kostant [5]. In § 6 we give Kostant's complete solution to which elements of the spherical principal series for Lorentz groups are irreducible.

2. The spherical principle series

Let G be a connected semisimple Lie group with finite center, $G = KAN$ an Iwasawa decomposition of G, K a maximal compact subgroup of G, AN an Iwaaswa subgroup of G, N the unipotent radical of AN, and A a maximal split torus of G acting semisimply on N. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$, be respectively the Lie algebras of G, K, A, N, and M the centralizer of A in K. Set $B = MAN$. Then $G/B = K/M$ under the map $kanB \mapsto kM$ for $k \in K, a \in A, n \in N$.

Let dx be the K-invariant normalized measure on K/M, and $\mathfrak{a}'$ and $\mathfrak{a}'_C$ respectively the spaces of real valued and complex valued linear forms on $\mathfrak{a}$. If $\lambda \in \mathfrak{a}'_C$, we define a Banach representation (π_λ, X^λ) of G as follows:

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