J. DIFFERENTIAL GEOMETRY
5 (1971) 333-340

REDUCTION OF THE CODIMENSION OF AN ISOMETRIC IMMERSION

JOSEPH ERBACHER

0. Introduction

Let $\psi: M^n \to \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of a connected n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\tilde{M}^{n+p}(\tilde{c})$ of constant sectional curvature $\tilde{c}$. When can we reduce the codimension of the immersion, i.e., when does there exist a proper totally geodesic submanifold N of $\tilde{M}^{n+p}(\tilde{c})$ such that $\psi(M^n) \subset N$? We prove the following:

Theorem. *If the first normal space $N_1(x)$ is invariant under parallel translation with respect to the connection in the normal bundle and l is the constant dimension of N_1, then there exists a totally geodesic submanifold N^{n+l} of $\tilde{M}^{n+p}(\tilde{c})$ of dimension $n + l$ such that $\psi(M^n) \subset N^{n+l}$.*

This theorem extends some results of Allendoerfer [2].

1. Notation and some formulas of Riemannian geometry

Let $\psi: M^n \to \tilde{M}^{n+p}(\tilde{c})$ be as in the introduction. For all local formulas we may consider ψ as an imbedding and thus identify $x \in M^n$ with $\psi(x) \in \tilde{M}^{n+p}$. The tangent space $T_x(M^n)$ is identified with a subspace of the tangent space $T_x(\tilde{M}^{n+p})$. The normal space $T_x^\perp$ is the subspace of $T_x(\tilde{M}^{n+p})$ consisting of all $X \in T_x(\tilde{M}^{n+p})$ which are orthogonal to $T_x(M^n)$ with respect to the Riemannian metric g. Let ∇ (respectively $\tilde{\nabla}$) denote the covariant differentiation in M^n (respectively $\tilde{M}^{n+p}$), and D the covariant differentiation in the normal bundle. We will refer to ∇ as the tangential connection and D as the normal connection.

With each $\xi \in T_x^\perp$ is associated a linear transformation of $T_x(M^n)$ in the following way. Extend ξ to a normal vector field defined in a neighborhood of x and define $-A_\xi X$ to be the tangential component of $\tilde{\nabla}_X \xi$ for $X \in T_x(M^n)$. $A_\xi X$ depends only on ξ at x and X. Given an orthonormal basis $\xi_1, \cdots, \xi_p$ of $T_x^\perp$ we write $A_\alpha = A_{\xi_\alpha}$ and call the A_α's the second fundamental forms associated with $\xi_1, \cdots, \xi_p$. If $\xi_1, \cdots, \xi_p$ are now orthonormal normal vector fields in a neighborhood U of x, they determine normal connection forms $s_{\alpha\beta}$ in U by

$$D_X \xi_\alpha = \sum_\beta s_{\alpha\beta}(X) \xi_\beta$$

Received May 20, 1970. This paper is a part of the author's doctoral dissertation written under the direction of Professor K. Nomizu at Brown University. The research was partially supported by the National Science Foundation.