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RIEMANNIAN STRUCTURES OF PRESCRIBED GAUSSIAN CURVATURE FOR COMPACT 2-MANIFOLDS

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Let (M, g) denote a smooth (say C^3) compact two-dimensional manifold, equipped with some Riemannian metric g. Then, as is well-known, M admits a metric g_c of constant Gaussian curvature c; in fact the metrics g and g_c can be chosen to be conformally equivalent. Here, we determine sufficient conditions for a given non-simply connected manifold M to admit a Riemannian structure $\bar{g}$ (conformally equivalent to g) with arbitrarily prescribed (Hölder continuous) Gaussian curvature $K(x)$. If the Euler-Poincaré characteristic $\chi(M)$ of M is negative, the sufficient condition we obtain is that $K(x) < 0$ over M. Note that this condition is independent of g, and this result is obtained by solving an isoperimetric variational problem for $\bar{g}$. If $K(x)$ is of variable sign for $\chi(M) < 0$, or if $\chi(M) > 0$, then the desired critical point may not be an absolute minimum and our methods do not succeed. If $\chi(M) = 0$, our methods apply when $K(x)$ satisfies an integral condition with respect to the given metric g (see § 3); this result is perhaps not unreasonable since, for $\chi(\mathrm{M}) \leq 0$, distinct Riemannian structures on M need not be conformally equivalent.

1. Preliminaries

By passing (if necessary) to the orientable two-sheeted covering space of M, we may suppose M is orientable and admits a Riemannian structure with metric tensor g, Gaussian curvature $k(x)$, and volume element dV. If $K(x)$ is a given (Hölder continuous) function defined on M, we shall attempt to determine a metric tensor $\bar{g}$, conformal with g, whose Gaussian curvature $\bar{k}(x) = K(x)$ at each point of M, i.e., we shall seek a smooth function σ defined on M such that $\bar{g} = e^{2\sigma}g$ and $\bar{k}(x) = K(x)$. To find the equation which will determine σ in terms of the given data $K(x)$, $k(x)$ and g, we recall that in terms of isothermal parameters (u, v) on M an element of arc length can be written $ds^2 = \gamma\{du^2 + dv^2\}$, and the Gaussian curvature can be written

$$(1) \qquad k = -\tfrac{1}{2}\gamma^{-1}\{(\log\gamma)_{uu} + (\log\gamma)_{vv}\}\,.$$

Setting $\gamma' = \gamma\exp 2\sigma$, in place of γ in (1), we obtain the desired equation

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