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QUARTIC STRUCTURES ON SPHERES

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1. Introduction

A C^∞ tensor field f of type $(1,1)$ on a connected C^∞ manifold P is said to define a polynomial structure of degree d if d is the smallest integer for which the powers $I, f, \cdots, f^d$ are dependent, and f has constant rank on P, where I is the identity transformation field. An almost complex manifold is a polynomial structure of degree 2. In the odd dimensional case, the almost contact manifolds provide examples of polynomial structures of degree 3. More generally, a globally framed f-manifold is a polynomial structure of degree 3. These are almost product spaces. In addition, almost product spaces are a source of further examples of polynomial structures [8].

The affine spaces R^{2n} and R^{2n-1} may be endowed with almost complex and almost contact structures, respectively, so these give the simplest examples of the manifolds considered the former having rank $2n$ and the latter rank $2n-2$. On the other hand, an odd dimensional sphere S^{2n-1} carries an almost contact structure, so it is a polynomial manifold which is globally framed. However, the even dimensional spheres are not almost complex except in dimensions 2 and 6, and whereas the contact structure on S^{2n-1} is "integrable", it is not even known whether S^6 can be given an almost complex structure which comes from a complex structure.

In a previous work [6], polynomial structures f of degree 4 were introduced and examples of them provided. These were of two types, namely,

$$f^4 + f^2 = 0 , \qquad (f^2 + I)^2 = 0 ,$$

the first one having rank $2n-1$ and the second maximal rank $2n$. Moreover, the former is globally framed and the latter is not. We show below that, except for a set of measure zero, the even dimensional spheres may be endowed with a quartic structure f, depending on a parameter λ, that is,

$$(f^2 + \lambda^2 I)(f^2 + I) = 0 , \qquad 0 < |\lambda| \leq 1 ,$$

and

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