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LIE TRANSFORMATION GROUPS OF BANACH MANIFOLDS

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Introduction

Let M be a Banach manifold which is not assumed to be Hausdorff, and let D denote the group of diffeomorphisms of M and $\mathbf{V}$ the Lie algebra of vector fields on M. A Lie group $\mathscr{G}$ is called a *Lie transformation group* of M if the underlying group G of $\mathscr{G}$ is a subgroup of D and the natural map $\alpha\colon (g, p) \mapsto g(p)$ from $\mathscr{G} \times M$ into M is a morphism (of manifolds). In this case, α induces a homomorphism α^+ from the Lie algebra $L(\mathscr{G})$ of $\mathscr{G}$ into $\mathbf{V}$ (cf. § 3). Conversely, we prove that the set of complete vector fields of a finite-dimensional subalgebra of $\mathbf{V}$ is a subalgebra (Proposition 8), and if $\mathbf{L}$ is a complete finite-dimensional subalgebra of $\mathbf{V}$ then there exists a unique connected Lie transformation group $\mathscr{G}$ such that α^+ is an isomorphism from $L(\mathscr{G})$ onto $\mathbf{L}$ (Theorem 9). In case M is finite-dimensional and Hausdorff, this result is due to Palais [4]. For the numerous applications in differential geometry, the reader is referred to [1]. Unfortunately, the proof of the just-mentioned special case given in [1] seems to be incomplete. The proof to be presented here is quite elementary; it relies heavily on the use of one-parameter families of diffeomorphisms, instead of one-parameter groups. To be more precise, we define a *curve in D* to be a morphism $\varphi\colon I_\varphi \times M \to M$ such that

(i) I_φ is an open interval in $\boldsymbol{R}$ containing 0;

(ii) the map $\varphi_t\colon p \mapsto \varphi(t, p)$ belongs to D, for all $t \in I_\varphi$;

(iii) $\varphi_0 = \mathrm{Id}_M$.

With φ we associate a time-dependent vector field $\delta\varphi$ by

$$\delta\varphi(t, p) = (\delta\varphi)_t(p) = (d/ds)_{s=t}\varphi_s(\varphi_t^{-1}(p)) \ .$$

The map $\varphi \mapsto \delta\varphi$ is injective (Proposition 4). The underlying group G of $\mathscr{G}$ turns out to be the set of diffeomorphisms φ_1 where φ is any curve in D such that $I_\varphi = \boldsymbol{R}$ and $(\delta\varphi)_t \in \mathbf{L}$ for all $t \in \boldsymbol{R}$. Using canonical coordinates of the second kind, G becomes a Lie group with the desired properties. We also prove the following criterion for a subgroup G of D to be a Lie transformation group (Theorem 10): assume there is a set S of curves in D such that $\{\varphi_t\colon \varphi \in S \text{ and } t \in I_\varphi\}$ generates G and that $\{(\delta\varphi)_t\colon \varphi \in S \text{ and } t \in I_\varphi\}$ generates a

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