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SPRAYS ON VECTOR BUNDLES

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1. Introduction

Suppose that $p\colon TX \to X$ is the tangent bundle of a smooth (C^∞) manifold X. A spray on X (or on the tangent bundle $p\colon TX \to X$), a notion due to Ambrose, Palais and Singer [1] is a smooth cross-section ξ of the tangent bundle $\sigma\colon TTX \to TX$ having the properties

$$p_*\xi = \sigma\xi\ , \qquad \xi\circ h_\alpha = h_\alpha(h_\alpha)_*\xi\ ,$$

where h_α is the smooth vector bundle morphism defined by scalar multiplication on each fiber by $\alpha \in R$ [4, p. 68], and $(h_\alpha)_*$ its tangent map.

The purpose of this paper is to generalize the concept of a spray on the tangent bundle of X to a spray on the bundle $q\colon TX \to X$ when TX admits an additional vector bundle structure q over X, and to discuss in some detail the case where $X = TM$, and M is a smooth manifold. We define sprays of the first and second type on an arbitrary vector bundle $q\colon TX \to X$, and in the case $X = TM$ show that each spray on M induces a spray of the second type on $\pi_*\colon TTM \to TM$, a spray of the first type on the tangent bundle ${}^1\pi\colon TTM \to TM$ of TM and investigate the relationship between these sprays. Sprays related to connections are investigated, and it is shown that the sprays of connections induced on the bundle structures of TTM by a linear connection ∇ on M coincide with the sprays induced on these bundles by the spray of the connection ∇.

The notation employed throughout the paper is essentially that of [4] and [5], with manifolds and vector bundles modeled on Banach spaces.

2. The general definition

Suppose that $p\colon TX \to X$ and $q\colon TX \to X$ are two vector bundle structures on TX over X, and $\phi\colon TX \to TX$ is a vector bundle isomorphism such that $q\circ\phi = p$.

Definition. A smooth cross-section ξ of $\sigma\colon TTX \to TX$ is called a spray of the first type on $q\colon TX \to X$ if it satisfies the conditions:

i. $q_*\xi = \sigma\xi$,

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