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MANIFOLDS ADMITTING NO METRIC OF CONSTANT NEGATIVE CURVATURE

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Let M be a compact n-dimensional Riemannian manifold of strictly negative sectional curvature, $K(\pi) < 0$ for all 2-planes π. If M admits a Riemannian metric of constant negative curvature, its Pontryagin classes are zero and consequently, if M is orientable and of dimension $4k$, its index in the sense of Hirzebruch is zero. For every positive integer k, there exist compact complex analytic manifolds of real dimension $4k$, arbitrarily large index and sectional curvature $-4 \leq K(\pi) \leq -1$. Such manifolds can admit no Riemannian metric of constant negative curvature.

This paper answers affirmatively a fundamental question: is the class of manifolds admitting Riemannian metrics of strictly negative sectional curvature larger than the class of manifolds admitting Riemannian metrics of constant negative sectional curvature ([7, p. 801])? In [6] Calabi asked a related question: Let M be a compact n-dimensional Riemannian manifold of strictly negative sectional curvature $K(\pi) < 0$. Can we find $\delta > 0$ sufficiently small so that if $-1 - \delta \leq K(\pi) \leq -1$ then M admits a Riemannian metric of constant negative sectional curvature? This paper does not resolve the question of Calabi but shows that $\delta < 3$ is necessary if the conjecture is true. For the rest of this paper see [2].

Definition 1. Let M be a connected, simply connected Riemannian manifold. A Clifford-Klein form of M is a Riemannian manifold M' whose simply connected Riemannian covering space is M.

The bounded symmetric domains M in C^n endowed with the Bergman metric are Riemannian symmetric spaces, and the group of complex analytic homeomorphisms of M contains the identity component of the isometries of M.

Definition 2. For a bounded symmetric domain M in C^n, a Clifford-Klein form M' of M is said to be complex analytic if it is a complex analytic manifold and if the natural map of M to M' is analytic. In [2] A. Borel proved the following:

Theorem 1. *A bounded symmetric domain always has a compact complex analytic Clifford-Klein form, and any such form has a regular finite Galois covering.*

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