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REGULARITY THEOREMS FOR PARTIAL DIFFERENTIAL OPERATORS

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1. In this paper we introduce the notion of a regular space and a regular linear or non-linear map. This is done in such a way as to abstract the notion of a space of smooth sections of a vector bundle and a linear or non-linear partial differential operator with smooth coefficients. The abstraction depends upon the notion of being able to take covariant derivatives of the sections as well as of the operators. This creates a category of spaces and maps, which is closed under composition and also inversion. These regular spaces, while being Fréchet spaces in one sense, have enough extra structures so that we may retain a number of the important theorems of the corresponding theory for Banach spaces. We prove for example that the set of regular linear maps with a regular inverse is open. We also prove an inverse function theorem: if a regular non-linear map has a derivative at a point which has a regular inverse, then the non-linear map has a regular inverse in a neighborhood of the point. It is hoped that the reader will find this a useful framework for passing from results for Banach spaces to results for smooth sections.

2. A regular space is a Fréchet space whose topology is defined in the following way by a norm and a finite collection of linear operators. Let E be a real (or complex) vector space with norm $\| \; \|$, and let $\nabla_1, \cdots, \nabla_N$ be a finite collection of linear operators which map E into itself and have closed graphs in the norm topology. If $I = (i_1, \cdots, i_k)$ is a multi-index of length $|I| = k$ with $1 \leq i_1, \cdots, i_k \leq N$, we define the higher-order operator $\nabla_I \colon E \to E$ as the composition

$$\nabla_I = \nabla_{i_1} \circ \nabla_{i_2} \circ \cdots \circ \nabla_{i_k} .$$

For each integer r we define the higher-order norm

$$\|f\|_r = \sum_{|I| \leq r} \frac{1}{|I|!} \|\nabla_I f\| .$$

Note that $\|f\|_0 = \|f\|$. Let $\mathscr{T}_r$ be the topology on E induced by the norm $\| \; \|_r$. If the topology $\mathscr{T}_\infty = \bigcup_{r=0}^{\infty} \mathscr{T}_r$ is complete (and hence Fréchet) we say that E, or more precisely the triple

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