J. DIFFERENTIAL GEOMETRY
4 (1970) 499-508

THE RIEMANNIAN STRUCTURE OF CERTAIN FUNCTION SPACE MANIFOLDS

WILLIAM B. GORDON

Introduction

In this paper we shall examine the properties of a certain class of projection and Green's operators which are associated with the tangent bundle of a Sobolev space $H^k(X, Y)$ (defined below) of maps from a manifold X to a manifold Y. In § 2 we use these results to describe a class of functions on $H^k(X, Y)$ which satisfy Condition C (in the sense of Palais and Smale). In § 3 we derive an expression for the riemannian sectional curvature of $H^k(X, Y)$. One might hope that the property of having a sectional curvature of definite sign would be transferred from Y to $H^k(X, Y)$. However, this is not the case. We shall construct examples of spaces Y whose riemannian curvatures are non-negative (zero, non-positive) such that the riemannian curvatures of $H^k(S^1, Y)$ are indefinite. (§ 3 does not depend on the results of § 2, and may be read immediately after § 1.)

1. A. Notation and basic definitions

Hereafter X and Y denote smooth finite dimensional riemannian manifolds, X compact and without boundary. We shall suppose that Y is isometrically and smoothly embedded in a euclidean space $\boldsymbol{R}^q$ (which we may always do by a well-known theorem of Nash).

We recall some basic facts in global analysis: (For general references see [1], [3], [4] or [5].) Let $\langle , \rangle$ denote the standard inner product on $\boldsymbol{R}^q$, $d\mu$ a smooth measure on X, k a positive integer, and A a strictly positive strongly elliptic self-adjoint operator (with smooth coefficients) of order $2k$ on $C^\infty(X, \boldsymbol{R}^q)$, say $A = 1 + \Delta^k$. Let $(u, v)_k = \int_X \langle Au, v \rangle d\mu$, and let $\|\cdot\|_k$ denote the corresponding norm. Two such operators A give rise to equivalent norms, and $H^k(X, \boldsymbol{R}^q)$ is defined to be the completion of $C^\infty(X, \boldsymbol{R}^q)$ with respect to $\|\cdot\|_k$. For $k = 0$, set $A = I$. By a theorem of Rellich, for $k < l$, the natural injection $H^l(X, \boldsymbol{R}^q) \to H^k(X, \boldsymbol{R}^q)$ is dense and compact. A theorem of Sobolev asserts that the $\|\cdot\|_k$ topology is larger than the C^t topology when $k > \frac{1}{2}di(X) + t$. Hence when $2k > di(X)$ the elements of $H^k(X, \boldsymbol{R}^q)$ are continuous maps and one may define

Communicated by R. S. Palais, September 12, 1969. This research was partially supported by NSF grants while the author was at Johns Hopkins University.