J. DIFFERENTIAL GEOMETRY
4 (1970) 487-497

EXTRINSIC RIGIDITY THEOREMS FOR COMPACT SUBMANIFOLDS OF THE SPHERE

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Introduction

In this paper we consider immersions $X: M \to S^m$ of a compact oriented Riemannian n-manifold M into the standard unit m-sphere S^m, and wish to find conditions on X which imply that it is a standard isometric immersion of a constant curvature n-sphere into S^m, i.e., to find extrinsic rigidity theorems. Our principal tools are certain integral formulas.

In § 1 we briefly discuss the problem of finding interesting integral formulas. As an example we derive a simple integral formula relating the scalar curvature to infinitesmal conformal transformations.

In § 2 we derive some integral formulas for compact hypersurfaces of S^{n+1} (Theorem A) by means of a variant of Newton's formula, and use these integral formulas to prove our first rigidity theorem (Corollary A).

In § 3 we generalize the first two formulas of Theorem A to the case of arbitrary codimension (Theorem B) and then derive an improvement of a rigidity theorem, originally due to De Giorgi and Simons, for compact minimal submanifolds of the sphere whose normal spaces are close enough to each other (Theorem C).

In the appendix we prove a weaker form of Theorem C using the theory of elliptic partial differential equations (Theorem D).

Notation and conventions. The inner product and norm in the Euclidean space E^{m+1} of dimension $m + 1$ are denoted by $(\, , \,)$ and $| \quad |$, respectively. All manifolds are assumed to be connected, and all immersions to be isometric. We denote the directional derivative of an E^{m+1}-valued function f along a vector v by $\nabla_v f$, which means componentwise differentiation. If $e_1, \cdots, e_n$ form a frame field for M we shall at times use the notation $f_{,j}$ for $\nabla_{e_j} f$, particularly if f is a component of a tensor. In all sections but the appendix we follow the index convention $1 \leq i, j, k, l \leq n$.

The author wishes to thank his advisor, Professor Shiing-Shen Chern, for many helpful suggestions and kind encouragement during the preparation of his dissertation, of which this paper forms a part. He also wishes to thank Professors Robert Gardner and David Gilbarg for several informative conversations.

Received October 4, 1969. The research for this paper was partially supported by an NSF Fellowship.