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FLAT RIEMANNIAN MANIFOLDS

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Some years ago, Auslander and Szczarba [2] gave an example of a compact flat manifold having nontrivial Stiefel-Whitney classes. This was, I imagine, somewhat surprising in view of the fact that the (real) Pontryagin classes of any Riemannian manifold can always be expressed in terms of the curvature tensor and thus are trivial for flat manifolds. § 5 of this paper is devoted to more examples of this general sort—the general paucity of examples in this area being a great hindrance to any reasonable conjectures. The only difficulty here is finding nontrivial yet computable situations.

The Main Theorem 2.3 of this paper is a decomposition theorem of sorts for compact flat Riemannian manifolds, and has strong consequences for the study of Stiefel-Whitney classes of such manifolds. The theorem is in the same general spirit as that of Calabi [17], but seems technically unrelated.

Perhaps the simplest compact flat manifolds are flat tori—these are the ones with trivial holonomy group. In § 2 we describe the notion of a "flat toral extensions". Roughly speaking, this is a way of putting together one compact flat manifold and a flat torus to make a new flat manifold the dimension of which is the sum of the dimensions of its constituents. It is, more technically speaking, a fiber bundle over the manifold with a flat torus as fiber. The group of the bundle is a quotient of the holonomy group of the manifold and it acts on the flat torus isometrically. The tangent bundle of such a construct turns out to be induced from a bundle over the base space—a technical fact which we exploit to obtain information about characteristic classes of flat manifolds.

The main theorem states that under some conditions a compact flat manifold arises as a flat toral extension. The condition is stated in terms of the holonomy group of the manifold: to wit

There is associated with each finite group Φ a positive integer $n(\Phi)$ such that: if M is a compact flat manifold with holonomy group Φ, and $\dim M > n(\Phi)$, *then M is a flat toral extension of another flat manifold of dimension* $\leq n(\Phi)$.

It is convenient to say M is a Φ-manifold if it is compact, flat and its holonomy group is isomorphic to Φ. The theorem implies that all Φ-manifolds "are" flat toral extensions of a finite set of flat manifolds. Here "are" means

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