J. DIFFERENTIAL GEOMETRY
4 (1970) 359-366

AN UPPER BOUND TO THE SPECTRUM OF Δ ON A MANIFOLD OF NEGATIVE CURVATURE

H. P. MCKEAN

1. Introduction

The spectrum of the standard Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ acting in L^2 $(R^2, dxdy)$ is the whole left half-line $(-\infty, 0]$. By contrast, the spectrum of the Laplace operator $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ for the hyperbolic plane $H = R^1 \times (0, \infty)$, acting in the appropriate space $L^2(H, y^{-2}dxdy)$ reaches only up to $-1/4$.

The proof is easy. To see that the spectrum lies to the left of $-1/4$, take a compact function $f \in C^\infty(0, \infty)$, and conclude from

$$\frac{1}{4}\left(\int_0^\infty f^2 y^{-2} dy\right)^2 = \frac{1}{4}\left(\int_0^\infty y^{-1} df^2\right)^2 = \left(\int_0^\infty y^{-1} ff' dy\right)^2$$
$$\leq \int_0^\infty y^{-2} f^2 dy \int_0^\infty (f')^2 dy$$

that

$$\frac{1}{4}\int_0^\infty f^2 y^{-2} dy \leq \int_0^\infty (f')^2 dy = -\int_0^\infty f y^2 f'' y^{-2} dy \ .$$

This bound is applied to a compact function $f \in C^\infty(H)$ viewed as a function of $y > 0$ for fixed $x \in R^1$, and the result is integrated from $-\infty$ to ∞ with regard to x. This gives

$$\frac{1}{4}\int_H f^2 y^{-2} dxdy \leq -\int_H f \Delta f y^{-2} dxdy \ ,$$

proving that the spectrum of Δ lies to the left of $-1/4$.

To see that the spectrum fills up the half-line $(\infty, -1/4]$, we can use the fact that if l is hyperbolic distance from $\sqrt{-1}$, then the so-called conical function

$$f_{-1/2+\sqrt{-1}c}(\mathrm{ch}\, l) = \int_0^1 (\mathrm{ch}\, l + \mathrm{sh}\, l \sin 2\pi x)^{-1/2+\sqrt{-1}c} dx$$

Communicated by I. M. Singer, September 18, 1969.