J. DIFFERENTIAL GEOMETRY
4 (1970) 311-333

CONFORMAL TRANSFORMATIONS OF RIEMANNIAN MANIFOLDS

MORIO OBATA

Introduction

Let (M, g), or simply M, be a Riemannian n-manifold with Riemannian metric g; throughout this paper manifolds are always assumed to be connected and C^∞. For any C^∞-function ρ the Riemannian metric $g^* = e^{2\rho}g$ is said to be *conformal* or *conformally related* to g, and for constant ρ it is said to be *homothetic to* g.

Let h be a C^∞-mapping of (M, g) into another Riemannian manifold (M^*, g^*). If the Riemannian metric h^*g^* induced on M by h is conformal (homothetic) to the original metric g, then h is called a *conformal* (*homothetic*) *mapping* of (M, g) into (M^*, g^*). (Under a conformal mapping the angle between two vectors is preserved.) h remains to be conformal under any conformal changes of metrics on M and M^*. If h is a diffeomorphism, then h is called a *conformal diffeomorphism* or briefly a *conformorphism*, and (M, g) is said to be *conformally diffeomorphic* or briefly *conformorphic* to (M, g^*) through h. If h is a conformorphism of (M, g) onto itself, then h is called a *conformal transformation* of (M, g).

For a group G of conformal transformations of (M, g), if there exists a conformally related metric $g^* = e^{2\rho}g$ with respect to which G is a group of isometries, then G is said to be *inessential*, otherwise *essential*.

Let $C(M, g)$, or simply $C(M)$, be the group of the conformal transformations of (M, g), and $I(M, g)$, or simply $I(M)$, the group of the isometries of (M, g); they both are known to be Lie groups with respect to the compact-open topology. It is known [4] that any compact subgroup of $C(M)$ is inessential. Therefore a maximal compact subgroup of $C(M)$ may be considered as a subgroup of $I(M)$ by a suitable conformal change of metric. In particular, if M is compact, so is $I(M)$. Hence there is a conformally related Riemannian metric g^* such that the group $I(M, g^*)$ is a maximal compact subgroup of $C(M, g) = C(M, g^*)$. It follows that on a compact Riemannian manifold M, $C(M)$ is essential if and only if it is not compact.

In this paper, we are mainly concerned with a 1-parameter group of conformal transformations of a Riemannian manifold M, and so we may

Received March 8, 1969. Supported by an NSF Senior Foreign Scientist Fellowship at Lehigh University in 1968–1969.