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# NEARLY KÄHLER MANIFOLDS

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## 1. Introduction

Let $M$ be a $C^\infty$ almost Hermitian manifold with metric tensor $\langle\ ,\ \rangle$, Riemannian connection $\nabla$, and almost complex structure $J$. Denote by $\mathscr{F}(M)$ the real valued $C^\infty$ functions on $M$, and by $\mathscr{X}(M)$ the $C^\infty$ vector fields of $M$. Then $M$ is said to be a *nearly Kähler manifold* provided $\nabla_X(J)(X) = 0$ for all $X \in \mathscr{X}(M)$. Examples of nearly Kähler manifolds which are not Kählerian are $S^6$ (with the canonical almost complex structure and metric), and more generally $G/K$, where $G$ is a compact semisimple Lie group and $K$ is the fixed point set of an automorphism of $G$ of order 3 (see [20]). If $\dim M \leq 4$, then $M$ is Kählerian [12]. Thus we henceforth assume $\dim M \geq 6$.

A nearly Kähler manifold has the following property. Let $p \in M$ and let $\gamma$ be a (piecewise differentiable) loop at $p$. Denote by $\tau_\gamma$ the parallel translation along $\gamma$, and let $\pi$ be the holomorphic section of the tangent space of $M$ at $p$ which contains $\gamma'(0)$. Then there exists $g \in U(n)$ such that $\tau_\gamma | \pi = g | \pi$, where we regard $U(n)$ as the structure group of the tangent bundle of $M$. Conversely it is easy to see that any almost Hermitian manifold with this property is a nearly Kähler manifold. We say that $U(n)$ is a *weak holonomy group* of $M$. In a subsequent paper we shall investigate weak holonomy groups $G$ for which $G$ is transitive on some sphere. The most interesting situation occurs when $G = U(n)$.

We show in this paper that many well known theorems about the topology and geometry of Kähler manifolds can be generalized to nearly Kähler manifolds. The key fact is that the curvature operator $R_{XY}(X, Y \in \mathscr{X}(M))$ of a nearly Kähler manifold satisfies certain identities described in § 2. These formulas resemble the corresponding formulas for Kähler manifolds sufficiently for us to carry over the proofs with a few changes.

In § 3 we generalize some formulas of [6] and [9] about holomorphic curvature to nearly Kähler manifolds. Furthermore, we define and discuss the properties of a particularly nice class of nearly Kähler manifolds, namely those of *constant type*. Pinching of nearly Kähler manifolds is discussed in § 4.

We observe in § 5 that a *compact nearly Kähler manifold of positive holomorphic sectional curvature is simply connected*. Furthermore *a complete*

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