J. DIFFERENTIAL GEOMETRY
4 (1970) 257-282

INFINITE DIMENSIONAL PRIMITIVE LIE ALGEBRAS

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Introduction

Transitivity questions in differential geometry can often be reduced to problems involving a certain type of "topologized" Lie algebra. In [6] we developed a structure theory for such algebras analogous to the classical Jordan-Hölder theory for groups and rings. (See paragraphs 4 and 5 in § 2 below.) The building blocks of this theory are the primitive algebras. In this paper we will study these algebras in detail. In particular we will sketch the "Cartan classification theorem" for primitive algebras over an algebraically closed field. In its broad outlines, our proof follows Weisfeiler [19].

Weisfeiler's proof is based on a remarkable theorem of Kac about infinite dimensional graded Lie algebras [11]. We will show how this theorem can be deduced by a simple completion trick from the following theorem proved in § 3 below: *An infinite dimensional linearly compact Lie algebra possesses at most one primitive subalgebra.* This theorem can be proved without assuming that the base field is algebraically closed, and the proof requires relatively little machinery (mainly some elementary results from commutative algebra).

This paper is organized as follows: The first section is a compendium of standard results on primitivity (included mainly for motivation).

§ 2 is a review of the material in [6]. In § 3 we prove our main theorems on primitivity (modulo some results on characteristics which are proved in the appendix). In § 4 we discuss primitivity for graded algebras and prove two important lemmas (Lemmas 4.2 and 4.3). In §§ 5, 6 we prove the theorem of Kac alluded to above. The rest of the paper is a sketch of the Cartan classification theorem, the main idea of which is the Weisfeiler trick of associating with every primitive Lie algebra a graded Lie algebra with the property that the term of degree zero acts irreducibly on the term of degree -1.

The author would like to thank Martin Golubitsky for advice on the material in § 8 and Shlomo Sternberg for many helpful suggestions throughout.

1. Primitivity

Let G be a group and let S be a set on which G acts. Let "$\sim$" be an equivalence relation on S. We will say that "$\sim$" is invariant with respect to G

Communicated by S. Sternberg, September 24, 1969.