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GRID MANIFOLDS

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This paper is intended to correlate the work of several authors on Riemannian manifolds on which there are defined systems of parallel fields of tangent planes. The main facts in this area are the reducibility theorem of de Rham [8] and the fibring theorems of Walker [14]. In order to make the exposition as simple as possible, we introduce the concepts of *local-product* and *grid*. Our principal result includes a 'grid' version of de Rham's theorem.

1. Almost-product and local-product structures

Let M be a smooth (i.e., C^∞) m-manifold without boundary. An *almost-product-structure* (or *AP-structure*) *of type* $(m_1, \cdots, m_r)$ on M is a direct sum decomposition $\tau_M = \bigoplus_{i=1}^{r} \xi_i$ of the tangent bundle of M into smooth subbundles ξ_i of dimension $m_i > 0$. Equivalently, an AP-structure on M is an ordered set $(\phi_1, \cdots, \phi_r)$ of mutually transverse smooth fields ϕ_i of tangent m_i-planes on M, with $\sum m_i = m$ (see [16]). Thus the m_i-plane $\phi_i(x)$ is the fibre of ξ_i at $x \in M$.

If each tangent field ϕ_i of an AP-structure $\Phi = (\phi_1, \cdots, \phi_r)$ on M is integrable, then we call Φ a *local-product structure*[1] (or *LP-structure*) on M *of type* $(m_1, \cdots, m_r)$, where as above $\dim \phi_i = m_i$. Equivalently, an LP-structure is an ordered set $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_r)$ of mutually transverse smooth foliations $\mathscr{F}_i$ with foil-dimension $m_i > 0$, and with $\sum m_i = m$. We denote the foil of $\mathscr{F}_i$ through $x \in M$ by $F_i(x)$, and call the ordered set $F(x) = (F_1(x), \cdots, F_r(x))$ the *foil-sequence* of $\mathscr{F}$ at x.

If Φ is an AP-structure on M, then the pair (M, Φ) is called an *AP-manifold*. Likewise $(M, \mathscr{F})$ is an *LP-manifold*, where $\mathscr{F}$ is an LP-structure on M. Suppose then that (M, Φ) and (N, Ψ) are AP-manifolds of types $(m_1, \cdots, m_r)$ and $(n_1, \cdots, n_s)$ respectively. An *AP-morphism* from (M, Φ) to (M, Ψ) is a pair (f, ζ), where $f\colon M \to N$ is a smooth map, $\zeta\colon (1, \cdots, r) \to (1, \cdots, s)$ is order-preserving ((i.e., $i \le j \Rightarrow \zeta(i) \le \zeta(j)$), and for all $i = 1, \cdots, r$, and all $x \in M$, $Tf(\phi_i(x)) \subset \phi_j(f(x))$, where $j = \zeta(i)$. An LP-morphism from one LP-

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[1] This use of the term 'local-product structure' is slightly different from the usual one in that we do not require that arbitrary sums of the fields ϕ_i be integrable. Thus our concept might be more precisely termed a weak local product.