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THE DENSENESS OF COMPLETE RIEMANNIAN METRICS

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The purpose of this note is to expand a bit on a theorem of K. Nomizu and H. Ozeki. In [1], they proved that on any paracompact connected C^∞ manifold there is a complete Riemannian metric. In fact, it was shown that if M is a C^∞ manifold, T_pM is the tangent space of M at $p \in M$, $\xi, \eta \in T_pM$, and $g_p(\xi, \eta) = g_p = g$ is a Riemannian metric on M, then there is a C^∞ function f on M such that $fg = (fg)_p(\xi, \eta) = f(p)g_p(\xi, \eta)$ is a complete Riemannian metric on M. We intend to prove the following theorem.

Theorem. *Let M be a connected C^∞ manifold with Riemannian metric g. Then given a compact subset $K \subseteq M$, there is a complete Riemannian metric h on M such that $h|_K = g|_K$, where $h|_K$ denotes "h restricted to K".*

Corollary. *If the space of Riemannian metrics on a connected C^∞ manifold M is given the topology of convergence of all derivatives of order up to l on compact subsets of M, then the complete metrics form a dense subset of the space of all metrics. (This is true for each fixed l, $1 \leq l \leq \infty$.)*

We make the following remarks.

Remark 1. The result can clearly be extended to non-connected paracompact manifolds.

Remark 2. In case M is compact, the result is trivial.

Before proceeding with the proof of the theorem, for the convenience of the reader we review the proof of Nomizu-Ozeki. Assume g is not complete, and let $B_p(r)$ be the metric ball

$$B_p(r) = \{q \in M \mid \mu_g(p, q) \leq r\} ,$$

where μ_g is the metric on M arising from the Riemannian metric g. Further let

$$d(p) = \{\sup r \mid B_p(r) \text{ is compact}\} .$$

Then $d\colon M \to \boldsymbol{R}$ is a continuous real-valued function. It is easy to see that $d(p) > 0$ for all $p \in M$, and it is not difficult to show that there is a C^∞ function $\bar{f}\colon M \to \boldsymbol{R}^+$ such that $\bar{f}(p) < 1/d(p)$ for all $p \in M$. Let $f = (\bar{f})^2$. Then fg is the required complete Riemannian metric. The proof of the completeness of fg is not difficult and can be found in [1].

We now give the proof of our theorem. Let $M = \bigcup K_j$, where the K_j are compact, $K_j \subseteq \text{int}\,(K_{j+1})$, (int (N) denotes the interior of a subset N of M). If

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