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A GENERALIZATION OF PARALLELISM IN RIEMANNIAN GEOMETRY; THE C^∞ CASE

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1. Introduction

Let $g\colon N^p \to M^m$ be a smooth (C^∞ or C^ω) immersion of riemannian manifolds. It is not assumed that the immersion is isometric. A smooth vector bundle map $G\colon T(N) \to T(M)$ between the tangent bundles will be called a *tangent bundle isometry* (T. B. I.) *along* g provided that the fibers $T(N)(n) = N_n$ are mapped isometrically by G into the fibers $T(M)(g(n)) = M_{g(n)}$. More generally, let E be a euclidean vector bundle over N, F be a euclidean vector bundle over M, and $G\colon E \to F$; then G will be called a *vector bundle isometry along* g if G maps the fibers $E(n)$ isometrically into the fibers $F(g(n))$. Let ∇ be the covariant derivative on M, and let $G\colon T(N) \to T(M)$ be a T. B. I. along $g\colon N^p \to M^m$. The *normal bundle to* G is the $(m-p)$-dimensional vector bundle $G^\perp$ (over N) whose fiber over $n \in N$ is the orthogonal complement, $\perp G(N_n)$, to $G(N_n)$ in $M_{g(n)}$. The *second fundamental form of* G, $II_G\colon G^\perp \to \mathrm{Hom}\,(T(N), T(N))$ is a vector bundle map defined in the following manner. If $v \in \perp G(N_n)$ and $x, y \in N_n$, extend y to a vector field Y on N in some neighborhood of n and put

$$\langle II_G(v)x, y\rangle_n = -\langle \nabla_{dg(x)}G(Y), v\rangle_{g(n)} .$$

Since ∇ is a metric connection, the definition is independent of the choice of Y. A T.B.I. G is *parallel* if $\mathrm{trace}\circ II_G\colon G^\perp \to R$ is the zero function.

Three pieces of evidence in support of this terminology were given in [2].

First, suppose that $\gamma\colon (a, b) \to M$ is a smoothly immersed curve, and let $d/dt\colon (a, b) \to T(a, b)$ be the standard unit vector field on (a, b). Then the formula

$$G\left(\frac{d}{dt}(t)\right) = Y(t) , \qquad t \in (a, b) ,$$

establishes a bijective correspondence between the set of T.B.I.s G along γ and the set of unit vector fields Y along γ. Under this correspondence the parallel T.B.I.s are paired with the parallel unit vector fields.

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