J. DIFFERENTIAL GEOMETRY
4 (1970) 155-167

GEOMETRY OF MANIFOLDS WITH STRUCTURAL GROUP $\mathscr{U}(n) \times \mathscr{O}(s)$

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K. Yano [12], [13] has introduced the notion of an f-structure on a C^∞ manifold M^{2n+s}, i.e., a tensor field f of type (1, 1) and rank $2n$ satisfying $f^3 + f = 0$, the existence of which is equivalent to a reduction of the structural group of the tangent bundle to $\mathscr{U}(n) \times \mathscr{O}(s)$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. An f-structure with $s = 2$ has arisen in the study of hypersurfaces in almost contact spaces [3]; this structure has been studied further by S. I. Goldberg and K. Yano [4].

The purpose of the present paper is to introduce for manifolds with an f-structure the analogue of the Kaehler structure in the almost complex case and of the quasi-Sasakian structure [2] in the almost contact case, and to begin the study of the geometry of manifolds with such a structure. In § 1 we introduce the Kaehler anologue and its geometry and in § 2 we study f-sectional curvature. § 3 discusses principal toroidal bundles and § 4 generalizes the Hopf-fibration to give a canonical example of a manifold with an f-structure playing the role of complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry.

1. Let M^{2n+s} be a manifold with an f-structure of rank $2n$. If there exists on M^{2n+s} vector fields ξ_x, $x = 1, \cdots, s$ such that if η_x are dual 1-forms, then

$$(*) \qquad \begin{aligned} &\eta_x(\xi_y) = \delta_{xy} , \\ &f\xi_x = 0 , \quad \eta_x \circ f = 0 , \\ &f^2 = -I + \sum \xi_x \otimes \eta_x , \end{aligned}$$

we say that the f-structure has *complemented frames*. If M^{2n+s} has an f-structure with complemented frames, then there exists on M^{2n+s} a Riemannian metric g such that

$$g(X, Y) = g(fX, fY) + \sum \eta_x(X)\eta_x(Y) ,$$

where X, Y are vector fields on M^{2n+s} [13], and we say M^{2n+s} has a *metric f-structure*. Define the *fundamental 2-form F* by

$$F(X, Y) = g(X, fY) .$$