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OPERATORS ON ALMOST HERMITIAN MANIFOLDS

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Introduction

Recently C. C. Hsiung [1] showed, among other results on the realization of the complex Laplace-Beltrami operator $\square$ on an almost Hermitian space, that if for an almost Hermitian structure the relation $\square = \Delta/2$ holds for all forms of degrees 0 and 1, then the structure is Kaehlerian, where Δ denotes the real Laplace-Beltrami operator; this result was a conjecture for some time and is an improvement of a theorem by Kodaira-Spencer [2]. In the present paper, we point out that the two definitions of the operator $\square$ given by Kodaira-Spencer and C. C. Hsiung respectively are different, and extend the above result of Hsiung by showing that for an almost Hermitian structure if $\square$ according to either definition is real on all forms of degrees 0 and 1, then the structure is Kaehlerian[1].

Let $\underset{r,s}{\Pi}$ be the projection mapping onto the subspace composed of elements of type (r, s) (see § 1), and ∂ the skew-derivation of dgree 1, which coincides on functions with $\underset{1,0}{\Pi}\, d$ and satisfies the relation $\partial d + d\partial = 0$. Then the definition of the integrable condition of the almost complex structure is given by $\partial^2 = 0$, [2]. By investigating the real and imaginary parts of the operator ∂, we express an equivalent condition of $\partial^2 = 0$ in terms of some real operators, and give a condition on a real operator for an almost Hermitian structure to be Kaehlerian.

1. Definitions

Let M^n be a Riemannian space, denote its fundamental metric tensor by $g_{\lambda\mu}$, and put $g = \det|g_{\lambda\mu}|$. (In the following the Greek indices $\lambda, \mu, \nu, \cdots$ run from 1 to n, the dimension of the space.) Let $\varepsilon^{\mu_1 \cdots \mu_p}_{\lambda_1 \cdots \lambda_p}$ be the generalized Kronecker's delta, $\varepsilon_{\lambda_1 \cdots \lambda_n}$ stand for $\varepsilon^{1 \cdots n}_{\lambda_1 \cdots \lambda_n}$, and $\mathscr{F}^p$ be the algebra of differential p-forms on M^n. Then the exterior differentiation $d\colon \mathscr{F}^p \to \mathscr{F}^{p+1}$ and the adjoint

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[1] After this paper was written, Hsiung informed me that at the Summer Institute on Relativity and Differential Geometry sponsored by the American Mathematical Society and the National Science Foundation at the University of California at Santa Barbara in 1962 he had raised the question: If $\square$ for an almost Hermitian structure is real, is the structure Kaehlerian?