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REPRESENTATIONS OF COMPACT GROUPS AND MINIMAL IMMERSIONS INTO SPHERES

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1. Let G be a compact group, K a closed subgroup of G, and $C(M)$ the space of all real-valued continuous functions on the homogeneous space $M = G/K$. Then G has a natural action on $C(M)$ given by $g \cdot f(p) = f(g^{-1}p)$, $f \in C(M)$, $g \in G$, $p \in M$. Let V be a (necessarily finite-dimensional) invariant irreducible subspace of $C(M)$. Then V may be given an inner product $\langle\ ,\ \rangle$ by $\langle f, g\rangle = \int_M fg d\mu$, where the homogeneous measure $d\mu$ normalized in such a way that $\int_M d\mu = \dim V$; relative to $\langle\ ,\ \rangle$, G acts orthogonally on V.

Definition. We say that V satisfies condition A if $f_1, \cdots, f_r$ form an orthonormal basis of V (in particular, $r = \dim V$), whenever $f_1, \cdots, f_r$ are linearly independent in V and $\sum_{i=1}^r f_i^2(p) = 1$ for all $p \in M$.

In this paper, we are concerned with the following question: For which homogeneous spaces M is condition A satisfied for all invariant irreducible subspaces of $C(M)$?

We shall restrict ourselves to the simplest homogeneous spaces, namely, the simply connected homogeneous spaces G/K, where (G, K) is a symmetric pair of compact type. We recall that for such a pair, G is a compact, semi-simple Lie group with an involutive automorphism $s: G \to G$ which is such that K is left fixed by s, and K contains the component of the identity of the fixed point set of s. To ensure the simply connectedness of G/K, we assume further that G is connected, simply connected and that K is connected. In this situation, condition A is strangely rare. In fact, we prove the following:

Theorem 1. *Let $M = G/K$ be a homogeneous space such that (G, K) is a symmetric pair of compact type, G is connected and simply connected, and K is connected. Then condition A is satisfied for all invariant, irreducible subspaces of $C(M)$ if and only if M is the 2-dimensional sphere $S^2 = SU(2)/U(1)$.*

In § 2, we prove Proposition 1, which says that the invariant, irreducible subspaces of $C(S^2)$ satisfy condition A. In § 3, we prove Proposition 2, which

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