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TOTALLY GEODESIC MAPS

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Introduction

A C^∞ map $f: X \to Y$ of finite-dimensional connected C^∞ Riemannian manifolds is defined to be *totally geodesic* if for each geodesic x_t in X the image $f(x_t)$ is a geodesic in Y [1, p. 123]. An equivalent definition is that f is connection-preserving, or *affine*. The global structure of these maps is investigated in this paper.

§ 1 contains preliminary material relating to the fundamental form of a map [1, p. 123]. In § 2 a global factorization theorem is proved, namely, if X is complete, then each totally geodesic map factors into a Riemannian submersion [3], [4], [6] and an immersion, both totally geodesic. In § 3 the submersion part of this factorization is studied. It is shown that the nontrivial totally geodesic Riemannian submersions with X complete can be characterized as fibre bundles with integrable (flat) connection [2], having complete Riemannian metrics on base and fibre, the latter being invariant under the structural group.

The proof of the factorization theorem starts with the fact (shown in § 1) that kernels of affine maps are holonomy-invariant. This implies that totally geodesic maps have constant rank, whence a local factorization follows. The global factorization uses results of Reinhart [8] and of Kobayashi and Nomizu [5]. The proofs in § 3 depend on results of Hermann [4] and O'Neill [6].

The material of § 1 and Theorem 3.3 in § 3 were part of the author's study of the fundamental form in his thesis done under the guidance of Professor James Eells, whom the author wishes to thank in this respect.

It is assumed without further mention that everything below is C^∞, and manifolds are connected. "Totally geodesic" is also abbreviated as "t.g.".

1. The fundamental form of a map

The fundamental form of a map expresses the manner in which connections on two manifolds are related by a map of these manifolds. It vanishes iff the map is affine, and also furnishes a good means of establishing the equivalence of the above two definitions of totally geodesic maps.

The definition of the fundamental form is conveniently given in the following

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