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CONFORMAL CHANGES OF RIEMANNIAN METRICS

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0. Introduction

Let M be an n-dimensional differentiable connected Riemannian manifold with metric tensor g. Since we consider several Riemannian metrics on the same manifold M, we denote by (M, g) the Riemannian manifold M with metric tensor g. The Riemannian metric g defines, in the tangent space at each point of the manifold, the inner product $g(X, Y)$ of two vectors X and Y at the point and the angle θ between two vectors by $\cos \theta = g(X, Y)/[\sqrt{g(X, X)} \cdot \sqrt{g(Y, Y)}]$. Let there be given two metrics g and g^* on M. If the angles between two vectors with respect to g and g^* are always equal to each other at each point of the manifold, we say that g and g^* are *conformally related* or that g and g^* are *conformal* to each other. A necessary and sufficient condition that g and g^* of M be conformal to each other is that there exist a function ρ on M such that $g^* = e^{2\rho}g$. We call such a change of metric $g \to g^*$ a *conformal change* of Riemannian metric. Yamabe [21] proved

Theorem A. *For any Riemannian metric given on a compact C^∞ differentiable manifold of dimension $n \geq 3$, there always exists a Riemannian metric which is conformal to the given metric and whose slalar curvature is constant.*

So in the study of conformal properties of a compact M we can assume the scalar curvature of M to be constant.

In the above discussion, what has been changed is the Riemannian metric g at each point of the manifold M. We are now going to consider point transformations which induce a conformal change of metric of the manifold.

Let (M, g) and (M', g') be two Riemannian manifolds, and $f: M \to M'$ a diffeomorphism. Then $g^* = f^{-1}g'$ is a Riemannian metric on M. When g^* and g are conformally related, that is, when there exists a function ρ on M such that $g^* = e^{2\rho}g$, we call $f: (M, g) \to (M', g')$ a *conformal transformation*. In particular, if $\rho =$ constant, then f is called a *homothetic transformation* or a *homothety*; if $\rho = 0$, then f is called an *isometric transformation* or an *isometry*.

The group of all conformal (homothetic or isometric) transformations of (M, g) on itself is called a *conformal transformation* (*a homothetic transformation or an isometry*) *group* and is denoted by $C(M)$ ($H(M)$ or $I(M)$). We

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