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A CLASS OF SCHUBERT VARIETIES

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1. Introduction

Recently, the author proved that in a real, complex or quaternionic Grassmann manifold provided with an invariant metric, the minimum locus is a Schubert variety and the conjugate locus is the union of two Schubert varieties (cf. [7], [8]). The purpose of this paper is to study these Schubert varieties in detail.

Let F be the field R of real numbers, the field C of complex numbers, or the field H of real quaternions, F^{n+m} $(n \geq 1, m \geq 1)$ an $(n+m)$-dimensional left vector-space over F provided with a positive definite hermitian inner product, and $G_n(F^{n+m})$ the Grassmann manifold of n-planes in F^{n+m}. The Schubert varieties which we shall study are defined by

$$V_l = \{\boldsymbol{Z} \in G_n(F^{n+m}) \colon \dim(\boldsymbol{Z} \cap \boldsymbol{P}) \geq l\}\,,$$

where $\boldsymbol{P}$ is a fixed p-plane in $F^{n+m}, 0 < p < n+m$, and l is a nonnegative integer. It is easy to see that $V_l = G_n(F^{n+m})$ if $l = \max(0, p-m)$, and V_l is empty if $l > \min(n, p)$.

Let $W_l = V_l \backslash V_{l+1}$ and let k be an integer such that $\max(1, p-m+1) \leq k \leq \min(n, p)$. Then

$$V_k = W_k \cup W_{k+1} \cup \cdots \cup W_{\min(n,p)}\,.$$

Roughly speaking, our main result is:

V_k is the disjoint union of a Grassmann manifold $W_{\min(n,p)}$ (which reduces to a point if $p = n$) and $\min(n,p) - k$ "tensor" bundles $W_l (k \leq l \leq \min(n,p) - 1)$ whose base space is $G_l(F^p) \times G_{n-l}(F^{n+m-p})$, whose standard fiber is the tensor product $(F^{n-l})^* \otimes F^{p-l}$ of an $(n-l)$-dimensional right vector space and a $(p-l)$-dimensional left vector space, and whose group is the tensor product $GL(n-l, F) \otimes GL(p-l, F)$.

In § 2, we describe a covering of $G_n(F^{n+m})$ by coordinate neighbourhoods. In § 3, we prove that each V_l is a Schubert variety and obtain the local equations of V_l in a coordinate neighbourhood in $G_n(F^{n+m})$, which show that V_{l+1} is the singular locus of V_l. In § 4, we obtain a covering of the manifold W_l

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